The Datasaurus and Persistent Homology

Ryan Jensen

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Outline

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- Creating of the Datasaurus

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- Persistent Homology

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- Results for Datasaurus

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- Persistent Homology
- Results for Datasaurus
- Large Scale Geometry and Persistent Homology

The Datasaurus

Goal

Justin Matejka and George Fitzmaurice morphed the "Datasaurus" into various shapes, all of which have essentially the same statistics. Their point was to always visualize data. The goal is to use persistent homology to differentiate the morphed shapes.

Figure: https://www.autodeskresearch.com/publications/samestats

The Datasaurus

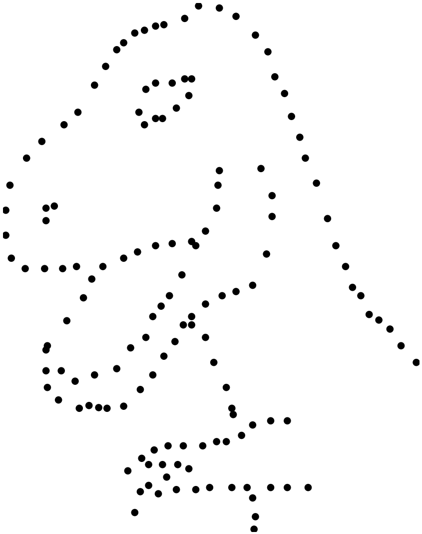

Figure: The Datasaurus

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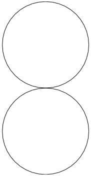
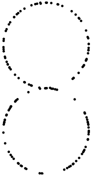


(a) Target Simplicial Complex

(b) Starting Point Cloud



(c) Ending Point Cloud

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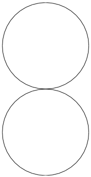

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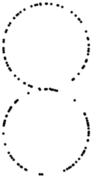

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Video Link

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- 100,000 iterations of moving points.

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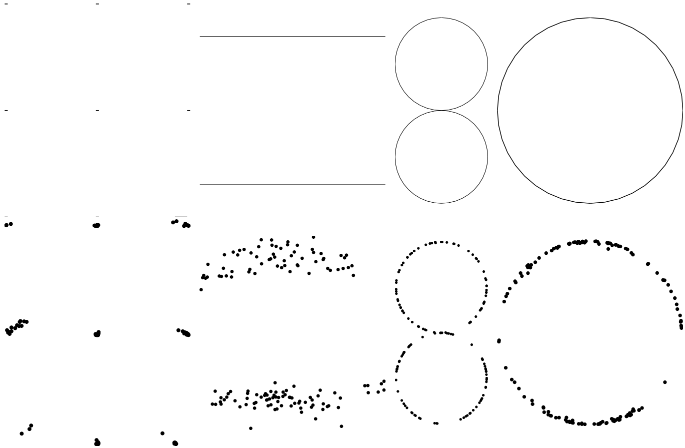

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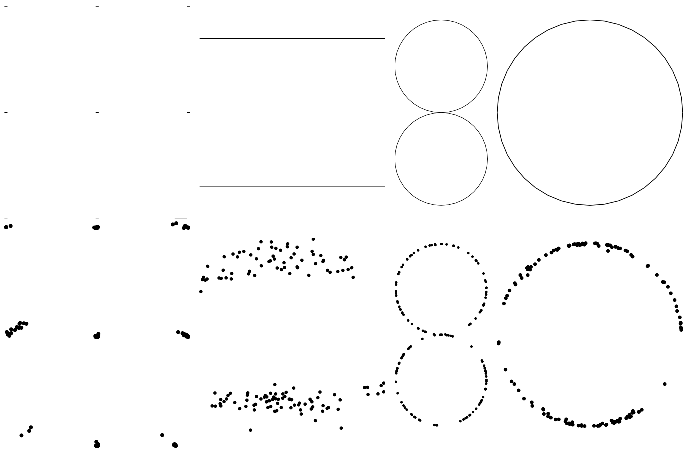

Video Link

Persistent Homology

Homology is First

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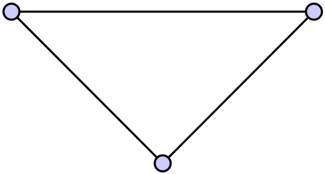

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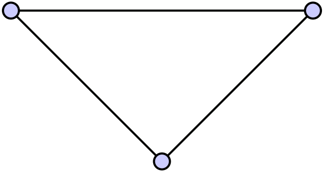

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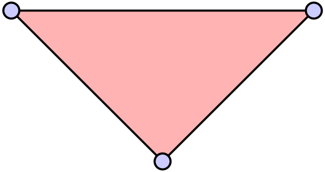

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- A **bar code** is a visual representation of the betti numbers.

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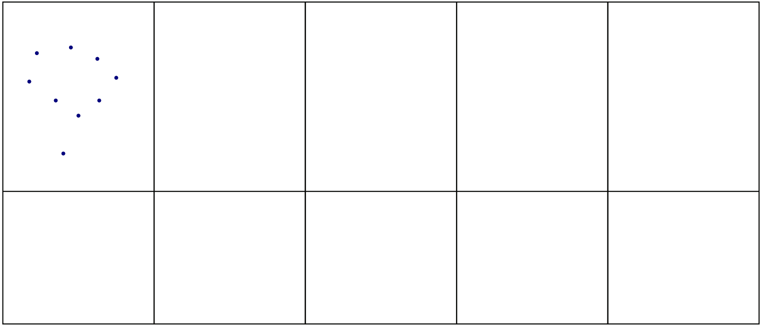
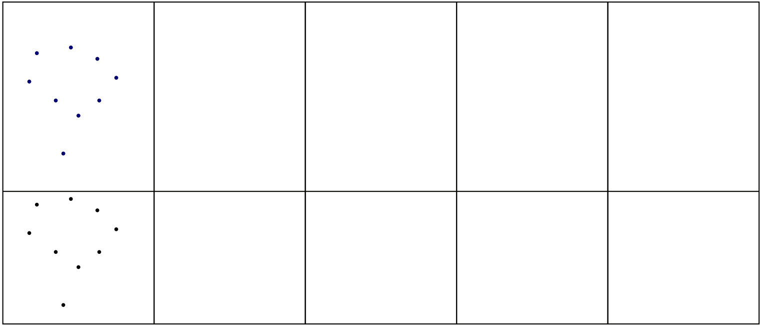
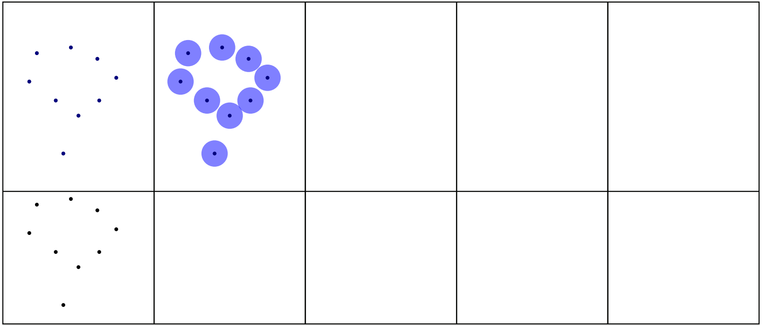
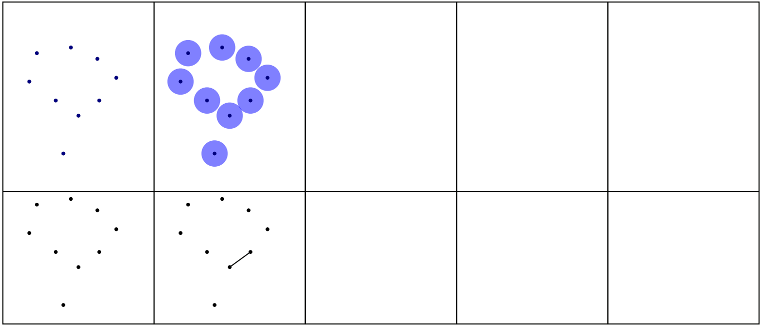
  - A homology class of $H_K(X_i)$ **dies** when its image is 0, otherwise it **persists**.









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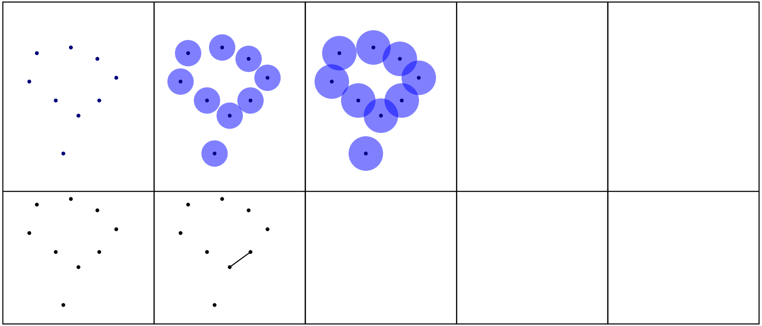

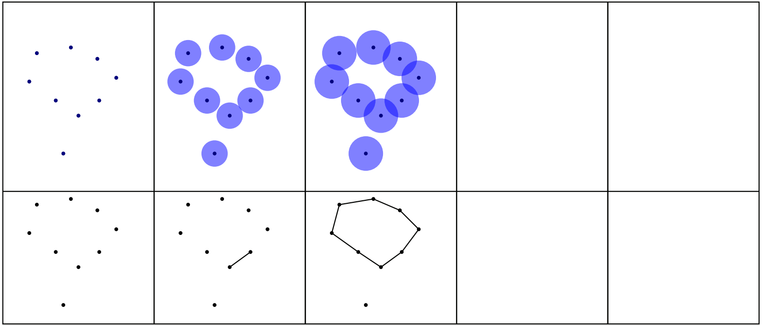

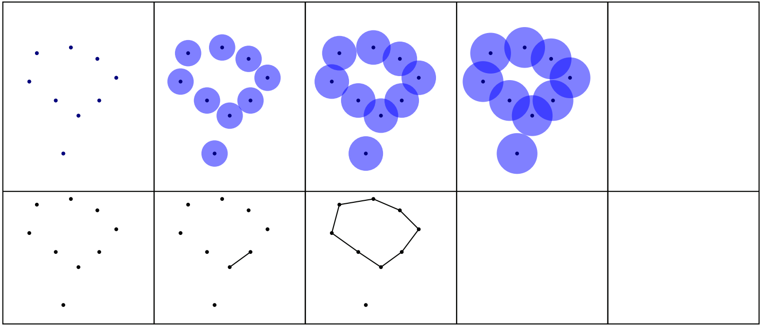

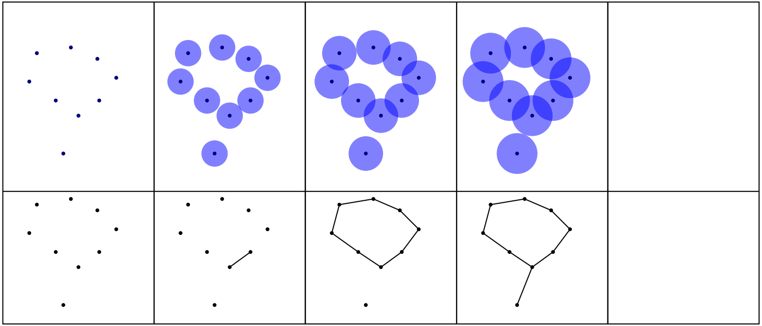

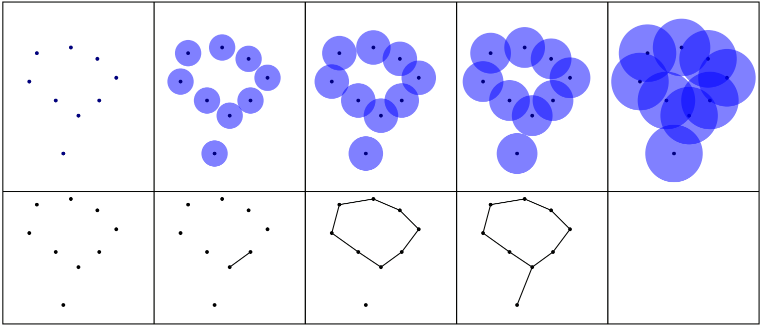

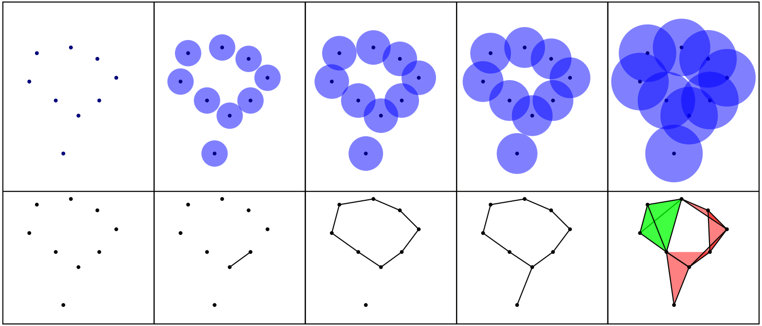

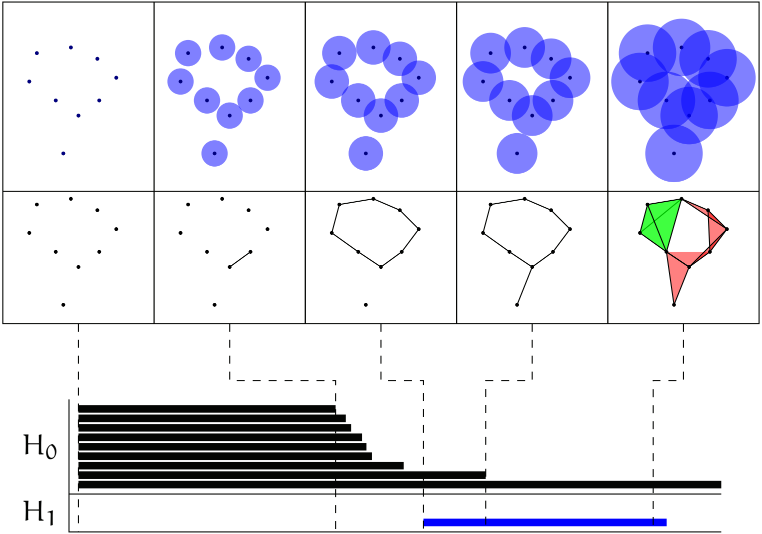
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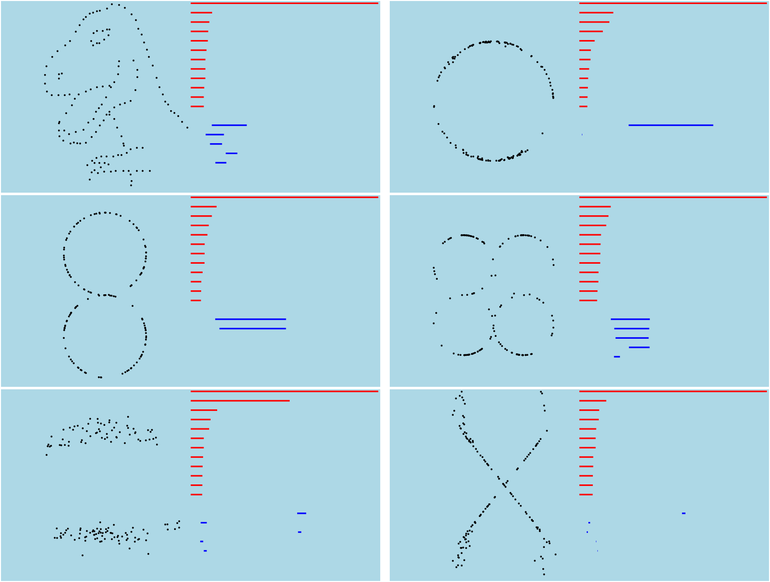
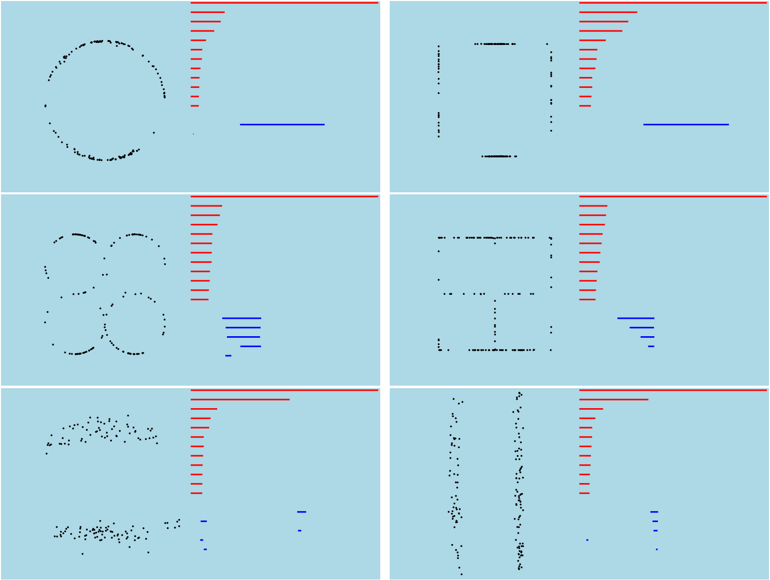
Persistent Homology

We can look at the persistent homology of the morphings and that may give us some idea that they are different.

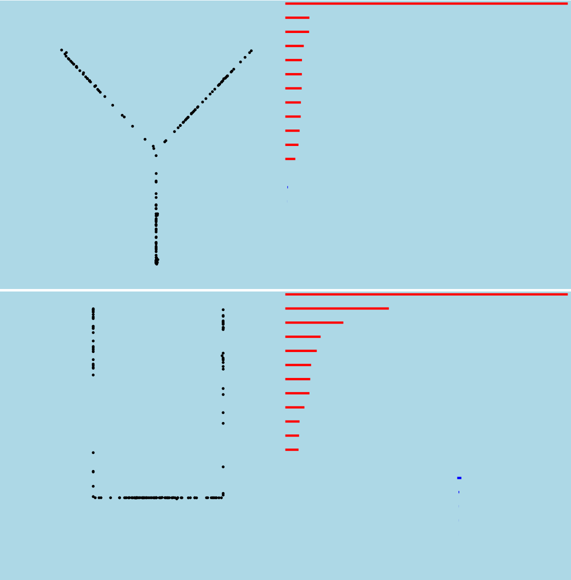
The Datasaurus

Video Link





The Y or The U?



Large Scale Geometry and Topology

Definition: Topology

A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X, called **open set**, so that:

0. $\emptyset, X \in \mathcal{T}$.
1. $U, V \in \mathcal{T}$ implies $U \cap V \in \mathcal{T}$.
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Definition: Large Scale Structure (Dydak & Hoffland 2006)

A **large scale structure** on a set X is a collection $\mathcal{LS} \neq \emptyset$ of families $\mathcal{B}$ of subsets of X, called **uniformly bounded** so that:

1. $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{LS}$ implies $\mathrm{St}(\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{LS}$.
2. $\mathcal{B}_1 \in \mathcal{LS}$ implies $\mathcal{B}_2 \in \mathcal{LS}$ if each element of $\mathcal{B}_2$ consisting of more than one point is contained in some element of $\mathcal{B}_1$.

Definition: Star

Let X be a set and $\mathcal{U}$ a collection of subsets of X, and $V \subset X$. The **star** of V with respect to $\mathcal{U}$, $\mathrm{St}(V, \mathcal{U})$, is the union of all elements of $\mathcal{U}$ which intersect V. That is

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The n-**star** of V with respect to $\mathcal{U}$ is defined as

$$\begin{aligned} \mathrm{St}^0(V, \mathcal{U}) &= V \\ \mathrm{St}^n(V, \mathcal{U}) &= \mathrm{St}\left(\mathrm{St}^{n-1}(V, \mathcal{U}), \mathcal{U}\right) \quad \text{for } n \geqslant 1. \end{aligned}$$

Distance via Stars

Staring gives a discrete “distance” (at least enough of a distance for a filtration used in persistent homology). Let $\mathcal{U}$ be a collection of subsets of a space X. For points x and y

$$d_{\mathcal{U}}(x, y) = \text{smallest } n \text{ so that } y \in \mathsf{St}^n(\{x\}, \mathcal{U}),$$

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If V and W are subsets of X, then

$$d_{\mathcal{U}}(V,W) = \text{smallest } n \text{ so that } W \cap \mathsf{St}^n(V,\mathcal{U}) \neq \emptyset,$$

or ∞ if no such n exits.

Example: Staring

Let $\mathbf{p}$ be a point in $\mathbb{R}^2$ and $V = \{\mathbf{p}\}$, and $\mathcal{U} = \{B(\mathbf{x}, 1)\}_{\mathbf{x} \in \mathbb{R}^2}$.

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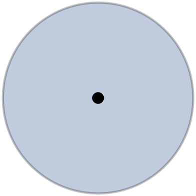
$$\mathrm{St}(V, \mathcal{U}) = B(\mathbf{p}, 2)$$



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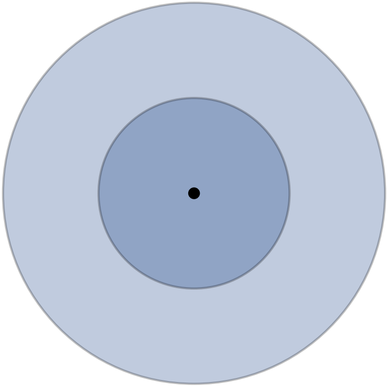
and

$$\mathrm{St}^2(V, \mathcal{U}) = B(\mathbf{p}, 4)$$



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  2. Arbitrary refinement corresponds to arbitrary unions.

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- Large scale geometry is (almost) the dual of topology.
  0. $\mathcal{LS} \neq \emptyset$ and the fact that point-sets cover a space correspond to $\emptyset, X \in \mathcal{T}$.
  1. Finite starrings (enlarging) covers corresponds to finite intersections.
  2. Arbitrary refinement corresponds to arbitrary unions.
- A metric space induces a large scale structure, just as it induces a topology. For example a large scale structure on $\mathbb{R}^n$ has uniformly bounded families $\mathcal{B}_r$ for $r \geqslant 0$, where

$$\mathcal{B}_r = \{B(\mathbf{x}, r)\}_{\mathbf{x} \in \mathbb{R}^n}.$$

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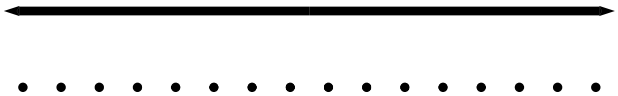

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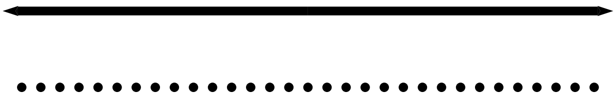

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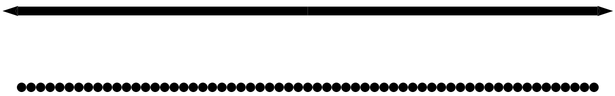

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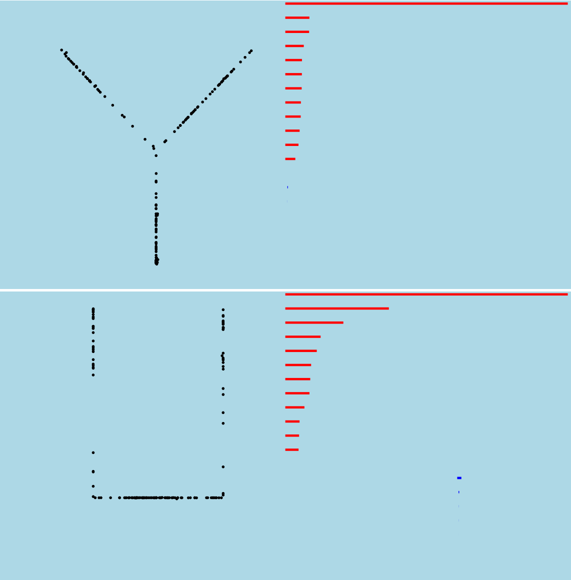

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Gunnar Carlsson developed the following algorithm, called mapper.

The Y or The U?

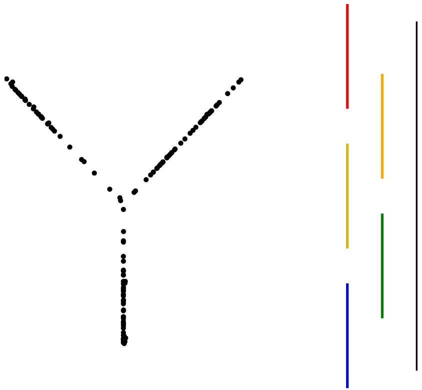

Figure: Height Function

The Y or The U?

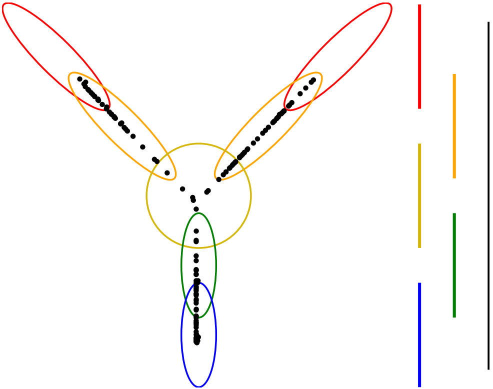

Figure: Clustering

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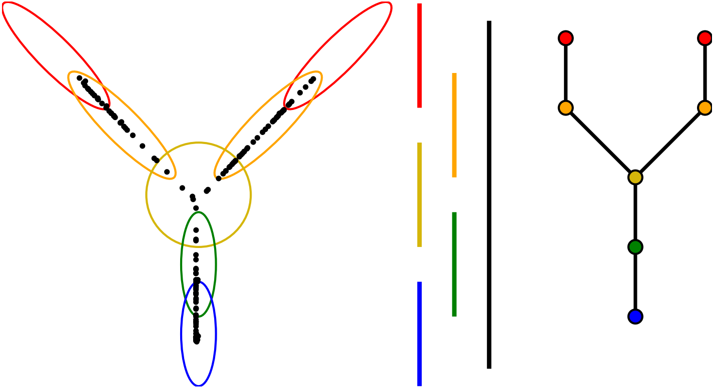

Figure: Graph

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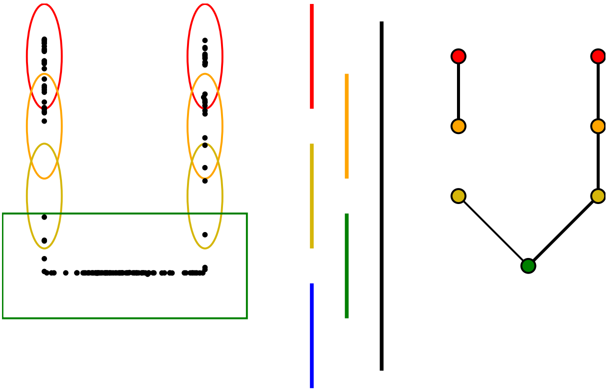

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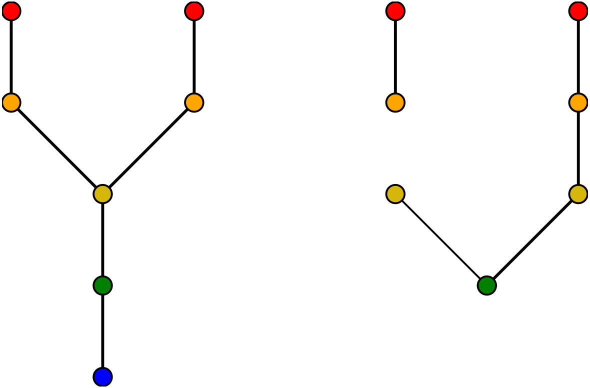

Figure: Graph Comparison

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- A graph can be produced as in the previous example.
- Cubical homology can also begin used, with a filtration by sub or super level sets.

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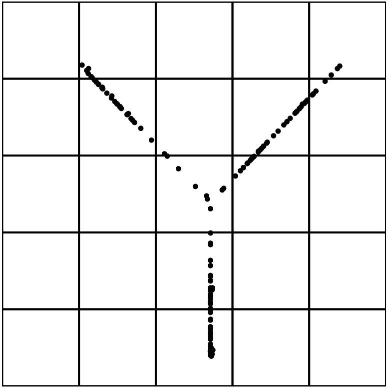

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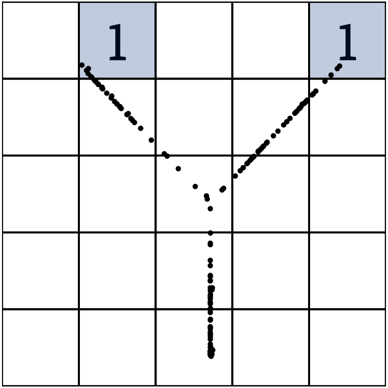

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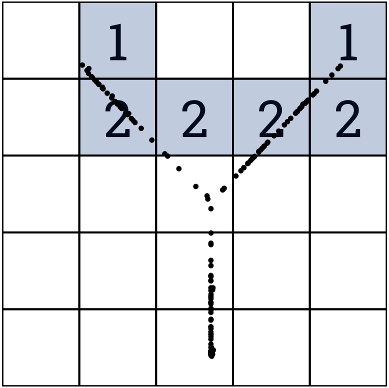

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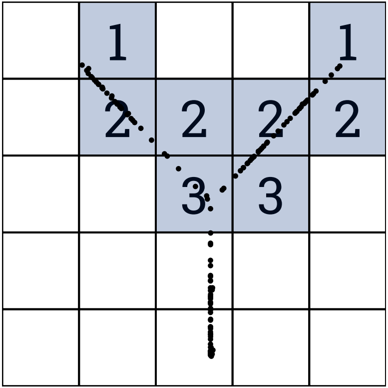

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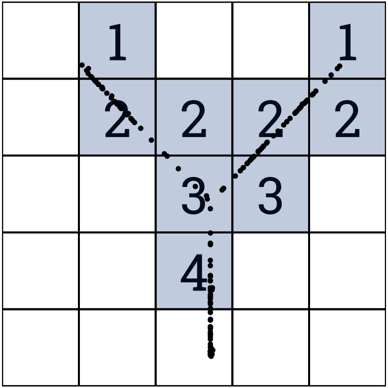

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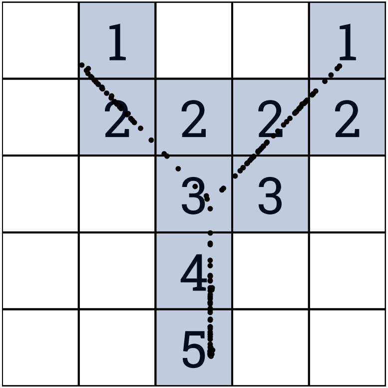

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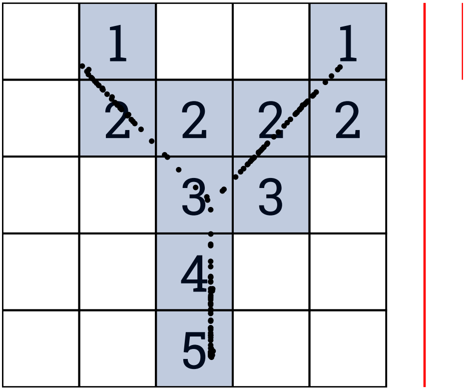

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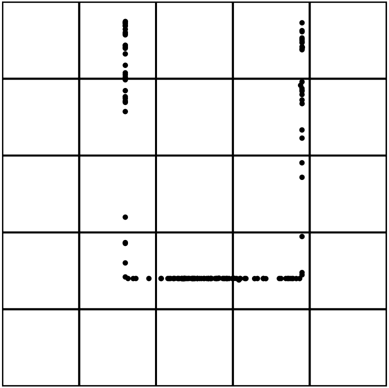

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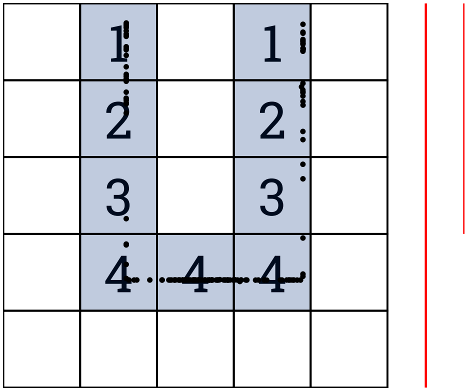

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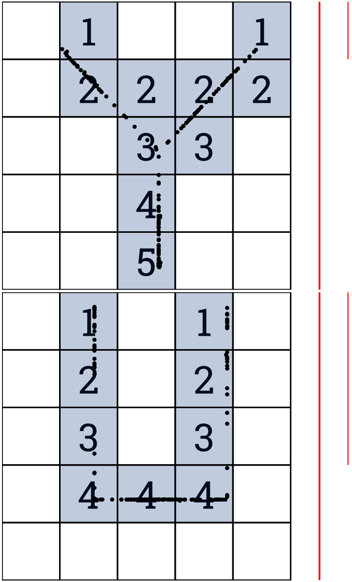

Thank You!