Math 233 Fall 2017-Jensen Practice Exam 1-A

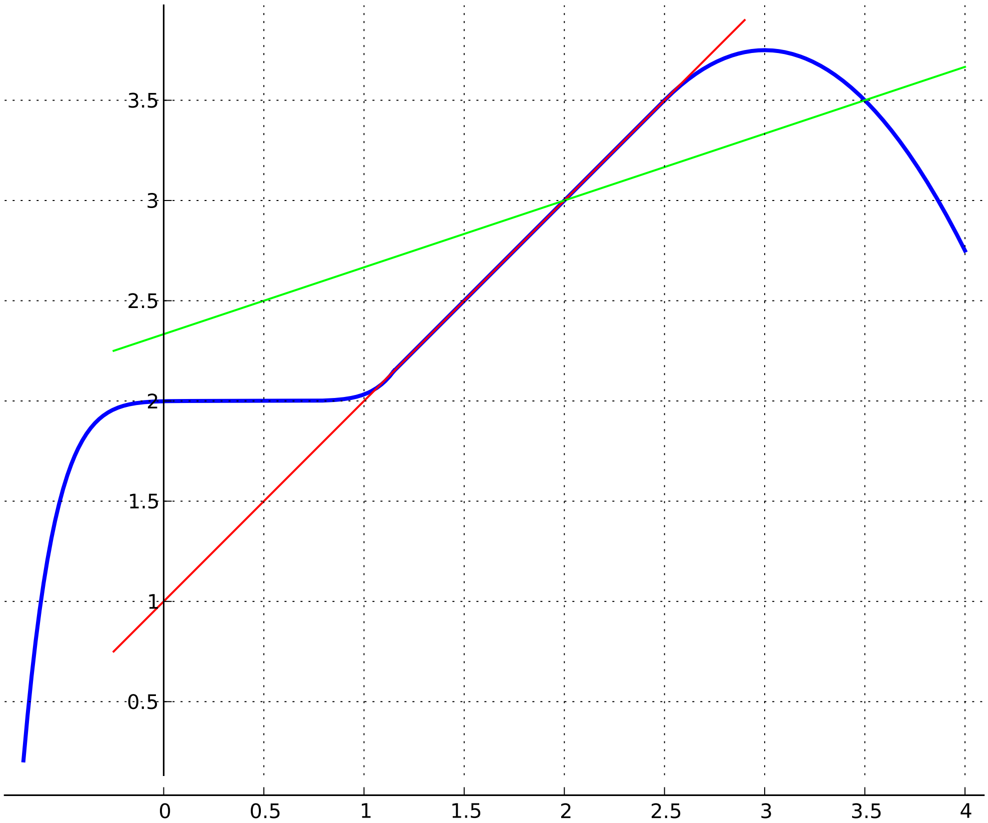
Name: ______________________________________________________________¡

**Directions**

- **Do not turn this page until told to do so.**
- Make sure your name is at the top.
- Answer all questions in the space provided. If you run out of room for an answer, continue on the back of the page.
- State any Laws or Theorems that you use.
- Show all your work in detail.
- You will have 1 hour to complete the exam.
- Good Luck!

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Total |
|---|---|---|---|---|---|---|---|---|---|---|
| Points: | 12 | 10 | 10 | 10 | 10 | 10 | 10 | 13 | 15 | 100 |
| Score: | | | | | | | | | | |

1. The graph of a function $f(x)$ is shown below.




(a) (4 points) Sketch the secant line to the graph for the interval [2,3.5].

(b) (4 points) Sketch the tangent line to the graph at the point $(2, 3)$.

(c) (4 points) Find the equation of the tangent line to the graph at the point $(2, 3)$.

**Solution:**

$$y = x + 1$$

2. A stone is tossed vertically into the air from ground level with an initial velocity of 15 m/s. Its height at time $t$ is $h(t) = 15t - 4.9t^2$ m.

(a) (8 points) Compute the stone's average velocity over the time intervals [1,1.01], [1,1.001], [.99,1], [.999,1].

**Solution:** The formula for average velocity is

$$\text{avg vel} = \frac{\Delta h}{\Delta t} = \frac{h(t_f) - h(t_i)}{t_f - t_i}.$$

Using our calculator and the above formula, we get

| Interval | Average Velocity |
|---|---|
| $[1, 1.01]$ | 5.151 |
| $[1, 1.001]$ | 5.195 |
| $[.99, 1]$ | 5.249 |
| $[.999, 1]$ | 5.205 |

(b) (2 points) Estimate the stone's instantaneous velocity at $t = 1$.

**Solution:** Using the table above, both the left and right limits approach 5.2. So the instantaneous velocity at $t = 1$ is 5.2 m/s.

3. (10 points) Use the Basic Limit Laws and the facts that for constants $c$ and $k$,

   1. $\lim_{x\to c} k = k$
   2. $\lim_{x\to c} x = c$

   to evaluate

$$\lim_{x\to 2} \sqrt{x^3 + 5x + 7}.$$

**Solution:**

$$\begin{aligned}
\lim_{x\to 2} \sqrt{x^3+5x+7} &= \sqrt{\lim_{x\to 2} x^3 + 5x + 7} && \text{By the Power/Roots Law} \\
&= \sqrt{\lim_{x\to 2} x^3 + \lim_{x\to 2} 5x + \lim_{x\to 2} 7} && \text{By the Sum Law} \\
&= \sqrt{\lim_{x\to 2} x^3 + 5\lim_{x\to 2} x + \lim_{x\to 2} 7} && \text{By the Constant Multiple Law} \\
&= \sqrt{\left(\lim_{x\to 2} x\right)^3 + 5\lim_{x\to 2} x + \lim_{x\to 2} 7} && \text{By the Power Law} \\
&= \sqrt{2^3 + 5(2) + 7)} && \text{By 1. and 2. of Problem 3.} \\
&= \sqrt{25} \\
\lim_{x\to 2} \sqrt{x^3+5x+7} &= 5
\end{aligned}$$

4. Let $g(x)$ be the function

$$g(x) = \begin{cases} \ln(-x-3) & \text{if } x < -3 \\ |1-x| & \text{if } -3 \le x < 2 \\ 4 & \text{if x=2} \\ 2\sqrt{2x} & \text{if } 2 < x \le 8 \\ 8 & \text{if } x > 8 \end{cases}$$

Determine whether the following are true or false.

(a) (2 points) $\lim_{x\to -3^+} g(x) = g(-3)$. T

(b) (2 points) $g(x)$ has a removable discontinuity at $x = -3$. F

(c) (1 point) $\lim_{x\to 2^-} g(x) = g(2)$. F

(d) (1 point) $g(8) = 8$. T

(e) (1 point) $\lim_{x \to 8} g(x)$ exists. T

(f) (1 point) $g(x)$ is continuous at $x = 8$. T

(g) (1 point) $g(x)$ has a jump discontinuity at $x = 8$. F

(h) (1 point) Bonus, write true. T

5. (a) (3 points) State the Squeeze Theorem.

**Solution:** If $l(x) \leq f(x) \leq u(x)$ for all $x$ in an open interval containing $c$ (except possibly at $c$ itself), and if

$$\lim_{x \to c} l(x) = L = \lim_{x \to c} u(x),$$

then $\lim_{x \to c} f(x) = L$.

(b) (7 points) Use the Squeeze Theorem to show that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.

**Solution:** We look at the interval $(-1, 1)$ which contains the point $c$. On this interval,

$$\begin{aligned} |\sin \frac{1}{x}| &\leq 1 \\ |x||\sin \frac{1}{x}| &\leq |x| \\ -|x| \leq x \sin \frac{1}{x} &\leq |x|. \end{aligned}$$

Since $\lim_{x \to 0} |x| = 0$ (NOTE: Actually we have not shown this, but this is from the book and the book assumes it. To show it evaluate the left and right hand limits.), and $\lim_{x \to 0} -|x| = -\lim_{x \to 0} |x| = 0$, we can use the Squeeze Theorem with $l(x) = -|x|$, and $u(x) = |x|$ to conclude that $\lim_{x \to 0} \sin \frac{1}{x} = 0$.

6. (a) (5 points) Find all vertical asymptotes of $\dfrac{x^2 - 4}{x^2 - 2x}$.

**Solution:** Generally vertical asymptotes happen when we have division by zero.

$$\begin{aligned}\lim_{x\to 0}\frac{x^2-4}{x^2-2x} &= \lim_{x\to 0}\frac{(x-2)(x+2)}{x(x-2)}\\ &= \lim_{x\to 0}\frac{x+2}{x}\\ &= \lim_{x\to 0}\frac{x}{x}+\lim_{x\to 0}\frac{2}{x} && \text{By the Limit Sum Law}\\ &= 1+\lim_{x\to 0}\frac{2}{x}\\ \lim_{x\to 0}\frac{x^2-4}{x^2-2x} &= \infty.\end{aligned}$$

So the line $x = 0$ is a vertical asymptote.

$$\begin{aligned}\lim_{x\to 2}\frac{x^2-4}{x^2-2x} &= \lim_{x\to 2}\frac{(x-2)(x+2)}{x(x-2)}\\ &= \lim_{x\to 2}\frac{x+2}{x}\\ &= \frac{2+2}{2} && \text{Since the reduced function is continuous}\\ \lim_{x\to 2}\frac{x^2-4}{x^2-2x} &= 2.\end{aligned}$$

So the line $y = 2$ is **not** a vertical asymptote.

(b) (5 points) Find all horizontal asymptotes of $\frac{2x^2+11x+12}{2x^2+x-3}$.

**Solution:** There is a horizontal asymptote at $y = 1$, since

$$\lim_{x\to\infty}\frac{2x^2+11x+12}{2x^2+x-3} = \frac{2}{2}\lim_{x\to\infty}x^{2-2} = 1.$$

Above we used a Theorem from the book. We also must check the limit as $x \to -\infty$. We do this one without the Theorem to illustrate another way of doing the problem.

$$\begin{aligned}\lim_{x\to-\infty}\frac{2x^2+11x+12}{2x^2+x-3} &= \lim_{x\to-\infty}\frac{2x^2+11x+12}{2x^2+x-3}\left(\frac{x^{-2}}{x^{-2}}\right)\\ &= \lim_{x\to-\infty}\frac{2+11x^{-1}+12x^{-2}}{2+x^{-1}-3x^{-2}}\\ &= \frac{\lim_{x\to-\infty}2+11x^{-1}+12x^{-2}}{\lim_{x\to-\infty}2+x^{-1}-3x^{-2}} && \text{Quotient Law}\\ &= \frac{2}{2} && \text{Sum, Const. Mult. Laws, and Thm}\\ \lim_{x\to-\infty}\frac{2x^2+11x+12}{2x^2+x-3} &= 1\end{aligned}$$

7. (10 points) Show that $\cos x + \frac{1}{2} = 2^x$ has at least one real root in the interval $[0, \pi]$.

**Solution:** Define the function $f(x) = \cos x + 1/2 - 2^x$. Since $\cos x$, $1/2$, and $2^x$ are everywhere continuous, $f(x)$ is continuous everywhere. In particular, it is continuous on $[0, \pi]$. Now notice that $f(0) = \cos 0 + 1/2 - 2^0 = 1 + 1/2 - 1 = 1/2 > 0$. Also $f(\pi) = \cos \pi + 1/2 - 2^\pi = -1 + 1/2 - 2^\pi < 0$. Then by using the Intermediate Value Theorem, there is a number $c \in [0, \pi]$ so that $0 = f(c) = \cos c + 1/2 - 2^c$. Thus $\cos c + 1/2 = 2^c$.

8. Find the limit if it exists. If necessary, state whether the limit is $\infty$, $-\infty$, or does not exist. Show your work.

(a) (5 points) $\lim_{x\to 4} \frac{16 - x^2}{x - 4}$.

**Solution:** We try direct substitution first, because it is easy and if it works it gives us the solution. At $x = 4$, we get

$$\frac{16 - 4^2}{4 - 4} = \frac{0}{0}.$$

This is in indeterminate form, so we do

$$\begin{aligned}\lim_{x\to 4} \frac{16 - x^2}{x - 4} &= \lim_{x\to 4} \frac{(4 - x)(4 + x)}{x - 4} \\ &= \lim_{x\to 4} \frac{(4 - x)(4 + x)}{-1(4 - x)} \\ &= \lim_{x\to 4} -(4 + x) \\ \lim_{x\to 4} \frac{16 - x^2}{x - 4} &= -8.\end{aligned}$$

The last equality comes by substitution, which we can do since the reduced function is continuous at $x = 4$.

(b) (8 points) $\lim_{x\to\infty} \sqrt{36x^2 - x} - 6x$.

**Solution:** Notice that this is in indeterminate form of $\infty - \infty$. So

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{36x^2 - x} - 6x &= \lim_{x\to\infty} \sqrt{36x^2 - x} - 6x \left(\frac{\sqrt{36x^2 - x} + 6x}{\sqrt{36x^2 - x} + 6x}\right) \\
&= \lim_{x\to\infty} \frac{(36x^2 - x) - 36x^2}{\sqrt{36x^2 - x} + 6x} \\
&= \lim_{x\to\infty} \frac{-x}{\sqrt{36x^2 - x} + 6x} \\
&= \lim_{x\to\infty} \frac{-x}{\sqrt{36x^2 - x} + 6x} \left(\frac{x^{-1}}{x^{-1}}\right) \\
&= \lim_{x\to\infty} \frac{-1}{\sqrt{36 - x^{-1}} + 6} \\
&= \frac{-1}{\sqrt{36} + 6} \qquad \text{Various Limit Laws} \\
\lim_{x\to\infty} \sqrt{36x^2 - x} - 6x &= -\frac{1}{12}
\end{aligned}$$

9. (15 points) Prove rigorously (using the formal definition of limit) that $\lim_{x\to 2} 4x - 1 = 7$.

**Solution:** Let $\epsilon > 0$ be given. Choose $\delta = \epsilon/4$. Then

$$\begin{aligned}
0 < |x - 2| &< \delta \\
|x - 2| &< \frac{\epsilon}{4} \\
4|x - 2| &< \epsilon \\
|4x - 8| &< \epsilon \\
|(4x - 1) - 7| &< \epsilon.
\end{aligned}$$

By the definition of limit, $\lim_{x\to 2} 4x - 1 = 7$. This is really all we need, but the trouble is choosing the correct $\delta$. In order to do this it is often convenient to work backwards. The following would be our scratch work:

$$\begin{aligned}
|f(x) - L| &< \epsilon \\
|(4x - 1) - 7| &< \epsilon \\
|4x - 8| &< \epsilon \\
4|x - 2| &< \epsilon \\
|x - 2| &< \frac{\epsilon}{4}.
\end{aligned}$$

So from that we know that $\delta \le \epsilon/4$ will work. We then do the forward direction as above.