10.4 - Absolute and Conditional Convergence

Def The series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

The series $\sum a_n$ converges conditionally if $\sum a_n$ converges, but $\sum |a_n|$ diverges.

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eg $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ converges absolutely, since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3} \text{ converges.}$$

Q|| Why does this converge?

A|| By the p-series test with $p = 3 > 1$.

Thm Absolute convergence implies convergence.

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

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pf Notice that $-|a_n| \leq a_n \leq |a_n|$, so

$$0 \leq |a_n| + a_n \leq 2|a_n|.$$

Notice that since $\sum |a_n|$ converges, so does

$$2\sum |a_n| = \sum 2|a_n|.$$

So $\sum(|a_n| + a_n)$ converges by the comparison test.

Finally $\sum a_n = \sum(|a_n| + a_n) - \sum |a_n|$

converges as well.

eg Again consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$,

This converges absolutely, per 1st example.

Thus $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ converges.

Thm Alternating Series Test

Let $\{b_n\}$ be a positive decreasing sequence converging to 0, i.e.

$$b_1 > b_2 > b_3 > \cdots > 0,$$

$$\lim_{n\to\infty} b_n = 0$$

then $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges. Also

$0 < S < b_1$, and $S_{2N} < S < S_{2N+1}$.

eg | Determine the convergence of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$.

Sol | $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$.

This diverges By the p-Series test with $p = 1/2 \leq 1$.

Thus $\sum \frac{(-1)^{n-1}}{\sqrt{n}}$ is not absolutely convergent.

Let $b_n = \frac{1}{\sqrt{n}}$. Notice that $\{b_n\}$ is a positive, decreasing sequence and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}}$

$= 0$.

Q2| A2| Thus by the alternating series test

$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = \sum_{n=1}^{\infty} \frac{(-1)}{\sqrt{n}}$ converges.

Thm | Let $S = \sum_{n=1}^{\infty} (-1) b_n$, where $\{b_n\}$ is a positive decreasing sequence converging to zero. Then

$$|S - S_N| \leq b_{N+1}.$$