Chapter 10

10.1 - Sequences

Rem

an interval of integers is the intersection of an interval I (real numbers) with $\mathbb{Z}$, ie

$$I \cap \mathbb{Z}$$

is an interval of integers.

eg

$[-4,3] \cap \mathbb{Z} = -4,-3,-2,-1,0,1,2,3$

$(0,4] = 1,2,3,4$

$(-\infty,\infty) = \mathbb{Z}$

Def | A *sequence* is a function a whose domain is an interval of integers. A sequence is *finite* if its domain is finite. A sequence is *infinite* if its domain is infinite.

Often we denote $a(n)$ as a_n, or the sequence a as $\{a_n\}$. The values a_n are called *terms* of the sequence. n is called the index.

Frequently we take the domain to be

$$[1, \infty) = 1, 2, 3, \ldots = \mathbb{P}$$

eg

$a : \mathbb{P} \to \mathbb{R}$

① $n \xmapsto{a} \frac{1}{n}$

② $a(n) = \frac{1}{n}$

③ $a_n = \frac{1}{n}$

④ $\{\frac{1}{n}\}$

all mean the same thing.

a_n is called the general term if a_n is given by a formula.

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

Def | Let L be a real number. The limit of a sequence $\{a_n\}$ is L, written as

$$\circledast \quad \lim_{n \to \infty} a_n = L,$$

if for all $\varepsilon > 0$, there is an M so that $|a_n - L| < \varepsilon$

when $n > M$.

If the limit of $\circledast$ exists, the sequence is said to converge. If the limit DNE, we say the sequence diverges. If the limit increases or decreases without bound, we say the limit diverges to ∞ or $-\infty$ respectively.

eg | 1, 2, 3, 4, ...

$\{n\}_{n \in \mathbb{P}}$

this diverges to ∞.

eg | -2, -4, -8, ...

$\{-2n\}_{n \in \mathbb{R}}$

Q!! T/F. This diverges to $-\infty$.

A!! T.

eg | Show that $\{\frac{1}{n}\}$ converges to 0.

$$\{\tfrac{1}{n}\} \longrightarrow 0$$

Sol | $L = 0$. Let $\varepsilon > 0$ be given

want: $\left|\frac{1}{n} - L\right| < \varepsilon$ when $n > M$ for some M

$$\left|\frac{1}{n} - 0\right| < \varepsilon$$

$$\left|\frac{1}{n}\right| < \varepsilon$$

$$\frac{1}{n} < \varepsilon$$

$$M = \frac{1}{\varepsilon} < n.$$

Choose $M = 1/\varepsilon$. Then when $M < n$

$$\left|\frac{1}{n} - L\right| < \varepsilon$$

$$\frac{1}{\varepsilon} < n$$

$$\frac{1}{n} < \varepsilon$$

$$\left|\frac{1}{n} - 0\right| < \varepsilon$$

eg

For $\varepsilon = \frac{1}{100}$, take $M = 1/\tfrac{1}{100} = 100$

when $n > M = 100$, say $n = 101$, then

$$100 < 101$$

$$\frac{1}{101} < \frac{1}{100} = \varepsilon$$

$$\left|\frac{1}{101} - 0\right| < \varepsilon$$

Thm If $\lim_{x\to\infty} f(x)$ exists, the the sequence

$a_n = f(n)$ converges to the same limit.

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x).$$

Eg Discuss the convergence of $a_n = \dfrac{n^2}{2^n - 1}$.

Define $f(x) = \dfrac{x^2}{2^x - 1}$. note that $a_n = f(n)$.

By Thm

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{2^x-1} \overset{LH}{=} \lim_{x\to\infty} \frac{2x}{\ln(2)\,2^x}$$

$$\overset{LH.}{=} \lim_{x\to\infty} \frac{2}{\ln^2(2)\,2^x} = 0.$$

Thm

Limit Laws for Sequences.

Assume that $\{a_n\}$ and $\{b_n\}$ are convergent, with $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$.

Then:

(i) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = L \pm M$

(ii) $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = L \cdot M$

(iii) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M}$, if $M \neq 0$.

(iv) $\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n = c \cdot L$

for any constant c.

Thm | Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be Sequences so that for some M

$$b_n \leq a_n \leq c_n$$

when $n > M$,

and $\lim_{n\to\infty} b_n = L = \lim_{n\to\infty} c_n$.

Then $\lim_{n\to\infty} a_n = L$.

Thm | If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

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pf | notice that $-|a_n| \leq a_n \leq |a_n|$ for all n.

also $\lim_{n\to\infty} -|a_n| = -\lim_{n\to\infty} |a_n| = -1(0) = 0 = \lim_{n\to\infty} |a_n|$

By Thm $\lim_{n\to\infty} a_n = 0$.

Thm Consider the sequence cr^n, for $c \neq 0$.

$$\lim_{n\to\infty} cr^n = \begin{cases} \text{diverges} & \text{if } r \leq -1 \\ 0 & \text{if } -1 < r < 1 \\ c & \text{if } r = 1 \\ \text{diverges} & \text{if } 1 < r \end{cases}$$

Pf. Suppose $r > 1$. Then $\lim_{n\to\infty} cr^n$

Define

$f(x) = |c| r^x$

Def A **recursive sequence** is a sequence where the first $n-1$ terms are given, and then the n^{th} term and beyond are computed using previous terms.

Def A **geometric sequence** is a recursive sequence of the form $a_0 = C$, $a_n = a_{n-1} \cdot r$, or equ.

$$\frac{a_{n+1}}{a_n} = r.$$

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eg $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \ldots$ is a geometric sequence.

with $a_0 = 3$ and $r = \frac{3/2}{3} = \frac{3(1/2)}{3} = 1/2$.

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eg $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$.

$0, 1, 1, 2, 3, 5, 8, 13, \ldots$

Thm

Consider the sequence cr^n for $c \neq 0$.

$$\lim_{n\to\infty} cr^n = \begin{cases} \text{diverges} & \text{if } r \leq -1 \\ 0 & \text{if } -1 < r < 1 \\ c & \text{if } r = 1 \\ \text{diverge} & \text{if } 1 < r. \end{cases}$$

Thm $\lim_{n \to \infty} \frac{R^n}{n!} = 0$ for all R.

Pf Assume $R > 0$, at first. Choose $M = \lfloor R \rfloor$ so that

$$M \leq R < M+1.$$

When $n > M$, we have

$$\frac{R^n}{n!} = \left[\left(\frac{R}{1}\right)\left(\frac{R}{2}\right) \cdots \left(\frac{R}{M}\right)\right]\left(\frac{R}{M+1}\right) \cdots \left(\frac{R}{n}\right)$$

Define $C = \left[\left(\frac{R}{1}\right)\left(\frac{R}{2}\right) \cdots \left(\frac{R}{M}\right)\right]$ and notice that $\frac{R}{M+1}$

Since $R < M+1$ then $\frac{R}{M+1} < 1$. Thus $\left(\frac{R}{M+1}\right) \cdots \left(\frac{R}{n}\right) < 1$

Then $\frac{R^n}{n!} \leq C\left(\frac{R}{N}\right)$ Then

$$0 \leq \frac{R^n}{n!} \leq C\left(\frac{R}{N}\right)$$

Thm $\lim_{n\to\infty} \frac{R^n}{n!} = 0$ for all R. $\quad n! = n(n-1)(n-2)\cdots 1$

pf. First assume $R > 0$. Choose $M = \lfloor R \rfloor$, so that

$$M \le R \le M+1.$$

when $n > M$, we have

$$\frac{R^n}{n!} = \left[\left(\frac{R}{1}\right)\left(\frac{R}{2}\right)\cdots\left(\frac{R}{M}\right)\right]\left(\frac{R}{M+1}\right)\left(\frac{R}{M+2}\right)\cdots\left(\frac{R}{n}\right)$$

Define $C = \left[\left(\frac{R}{1}\right)\cdots\left(\frac{R}{M}\right)\right]$. notice that

Since $R < M+1$. Then $\left(\frac{R}{M+1}\right)\left(\frac{R}{M+2}\right)\cdots\left(\frac{R}{n}\right) < 1$

$\frac{R}{M+1} < 1$

So $\frac{R^n}{n!} \le C\frac{R}{n}$ then $0 \le \frac{R^n}{n!} \le C\frac{R}{n}$

Pf

Since $\lim_{n\to\infty} \frac{R}{n} = 0$, the Squeeze thm Says that

$$\lim_{n\to\infty} \frac{R^n}{n!} = 0.$$

If $R=0$ this is clearly true

Q: If $R \geq 0$, then $\lim_{n\to\infty} \frac{R^n}{n!} = ?$

If $R<0$, then $\left|\frac{R^n}{n!}\right|$ is positive.

By 1st part $\lim_{n\to\infty} \left|\frac{R^n}{n!}\right| = 0$. Then

apply prev. Thm $\left(-\left|\frac{R^n}{n!}\right| \leq \frac{R^n}{n!} \leq \left|\frac{R^n}{n!}\right|\right)$

to get that $\lim_{n\to\infty} \frac{R^n}{n!} = 0$.

Pf | Since $\lim_{n\to\infty} C\frac{R}{n} = 0$, the Squeez thm Says

$$\lim_{n\to\infty} \frac{R^n}{n!} = 0.$$

Case 2: $R = 0$.

$\boxed{Q2}$ $\lim_{n\to\infty} \frac{R^n}{n!} = 0$.

Case 3: $R < 0$. Then $\left|\frac{R^n}{n!}\right|$ is positive,

By case 1, $\lim_{n\to\infty} \left|\frac{R^n}{n!}\right| = 0$

Then apply prev thm: $-\left|\frac{R^n}{n!}\right| \leq \frac{R^n}{n!} \leq \left|\frac{R^n}{n!}\right|$

to get $\lim_{n\to\infty} \frac{R^n}{n!} = 0$.

Thm | If f is continuous and $\lim_{n\to\infty} a_n = L$, then

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L).$$

eg | Suppose $f(x) = e^x$, and $a_n = \frac{3n}{n+1} = \frac{3}{1+\frac{1}{n}}$.

Notice that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3}{1+\frac{1}{n}} = 3$.

Then

$$\lim_{n\to\infty} e^{\frac{3n}{n+1}} = \lim_{n\to\infty} f\left(\frac{3n}{n+1}\right) = f\left(\lim_{n\to\infty} \frac{3n}{n+1}\right) = f(3) = e^3,$$

Def A sequence $\{a_n\}$ is __bounded above__ if there is an M so that $a_n \le M$ for any n. M is called an __upper bound__ of $\{a_n\}$.

A sequence $\{b_n\}$ is bound below if there is an m so that $m \le b_n$ for any n. m is called a lower bound of $\{b_n\}$.

A sequence $\{a_n\}$ is __bounded__ if it is bounded above and bounded below.

Thm If f is continuous and $\lim\limits_{n\to\infty} a_n = L$, then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).$$

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eg Suppose $f(x) = e^x$, and $a_n = \dfrac{3n}{n+1}$.

Notice that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{3}{1+\frac{1}{n}} = 3.$

$= f\left(\lim\limits_{n\to\infty}\left(\frac{3n}{n+1}\right)\right) = f(3)$

Then $$\lim_{n\to\infty} e^{\left(\frac{3n}{n+1}\right)} = \lim_{n\to\infty} f\left(\frac{3n}{n+1}\right) = f(3) = e^3.$$

Def. A sequence $\{a_n\}$ is <u>bdd above</u> if $\exists$ M so that $a_n \le M$ for all n. M is called an <u>upper bound</u> of $\{a_n\}$.

A sequence $\{a_n\}$ is <u>bdd below</u> if $\exists$ m so that $m \le a_n$ for all m. m is called a lower bound of $\{a_m\}$

A sequence $\{a_n\}$ is <u>bdd</u> if bdd above/below. [illegible]

Thm If $\{a_n\}$ converges, then $\{a_n\}$ is bounded.

Def ~~A sequence~~ Let $\{a_n\}$ be a sequence.

- $\{a_n\}$ is inc. if $a_n \le a_{n+1}$ for all n
- $\{a_n\}$ is dec. if $a_n \ge a_{n+1}$ for all n
- $\{a_n\}$ is monotone if it is either inc. or dec.

Thm If $\{a_n\}$ is inc. and $a_n \le M$ then $\{a_n\}$ converges and $\lim\limits_{n\to\infty} a_n \le M$.

If $\{a_n\}$ is dec. and $a_n \ge m$ then $\{a_n\}$ converges and $\lim\limits_{n\to\infty} a_n \ge m$.

Thm | If $\{a_n\}$ converges, the $\{a_n\}$ is bounded.

Def | Let $\{a_n\}$ be a sequence.

(i) $\{a_n\}$ is increasing if $a_n \leq a_{n+1}$ for all n.

(ii) $\{a_n\}$ is decreasing if $a_n \geq a_{n+1}$ for all n.

(iii) $\{a_n\}$ is monotonic if it is either inc. or dec.

Thm. | If $\{a_n\}$ is monotonic and bounded, then it converges.

eg | Find $\lim_{n \to \infty} a_n$, where $a_1 = \sqrt{2}$

$a_2 = \sqrt{2\sqrt{2}}$

$a_3 = \sqrt{2\sqrt{2\sqrt{2}}}$

$\vdots$

Sol | $\{a_n\}$ is bdd and inc. $\Rightarrow$ a limit exist.

Claim $\{a_n\}$ is bounded above by 2.

$2 < 4$

$a_1 = \sqrt{2} < 2$

If $a_K < 2$, then $a_{K+1} < 2$ Since

$$a_{K+1} = \sqrt{2\, a_K} < \sqrt{2 \cdot 2} = 2$$

So

$a_1 < 2 \Rightarrow a_2 < 2 \Rightarrow a_3 < 2 \cdots$

Claim $\{a_n\}$ is increasing.

$a_{k+1} = \sqrt{2a_k} > \sqrt{a_k a_k} = a_k$

$\uparrow$

Since $2 > a_k$

Thus by Thm $\{a_n\}$ is inc, bdd, so a limit exists.

$$L = \sqrt{2\sqrt{2\sqrt{2\cdots}}} = \sqrt{2L}$$

$$L = \sqrt{2L}$$

$$L^2 = 2L$$

$$L = 2$$

$\cancel{L=0}$ Sine $a_1 = \sqrt{2} > 0$ and $\{a_n\}$ is inc.