10.3 - Convergence of Series with Positive terms.

Def A positive series $\sum a_n$ is a series where each $a_n > 0$.

A non-negative series $\sum b_n$ is a series where each $b_n \geq 0$.

Thm If $S = \sum_{n=1}^{\infty} a_n$ is a non-negative series, then either

① $\{S_n\}$ is bound above and S converges, or

② $\{S_n\}$ is not bounded and S diverges.

pf S_n is monotonic.

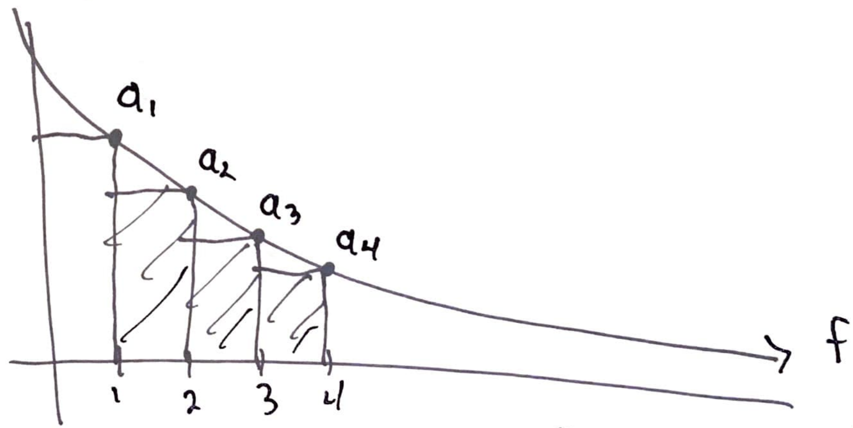
Thm | Integral Test

Let $a_n = f(n)$, where f is a positive, decreasing, continuous function for $x \geq 1$.

① If $\int_1^\infty f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

② If $\int_1^\infty f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

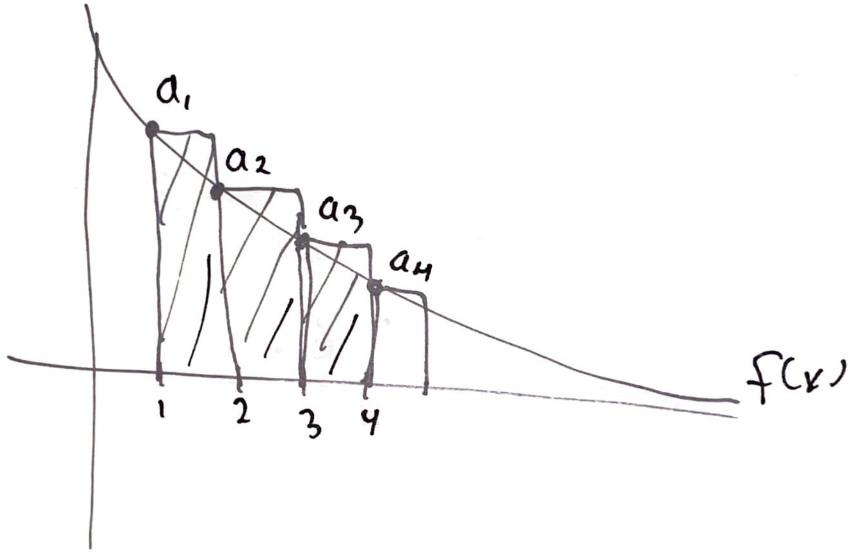
Pf. Suppose $\int_1^\infty f(x)\,dx$ converges.



$$a_2 + a_3 + a_4 + \cdots a_n \leq \int_1^n f(x)\,dx \leq \int_1^\infty f(x)\,dx = L$$

$$S_n \leq L + a_1$$

(2) Suppose $\int_1^\infty f(x)\,dx$ diverges



$$\int_1^n f(x)\,dx \leq \underline{a_1 + a_2 + \cdots + a_{n-1}}$$

If $\int_1^n f(x)\,dx \to \infty$ then $S_n \to \infty$

Q1| T/F : I reasoned out to myself why this is true.

eg | Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ = Harmonic Series diverges.

Sol | Define $f(x) = \frac{1}{x}$.

For $x \geq 1$:

- $f(x) > 0$
- $f(x)$ is decreasing
- $f(x)$ is continuous
- Also $\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x}\,dx = \infty$

Then $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges since $\frac{1}{n} = a_n = f(n)$,

by the integral test.

Thm Convergent p-Series

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, and diverges otherwise.

Pf. If $p \leq 0$, then $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$. So By the divergence test $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

If $p > 0$. Then define $f(x) = \frac{1}{x^p}$.

For $x \geq 1$: $f(x) > 0$

$f(x)$ is decreasing

$f(x)$ is continuous,

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p \leq 1 \end{cases}$$

By the p-integral test.

Thus since $\frac{1}{n^p} = a_n = f(n)$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 \leq p < 1$ By Integral test.

eg Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Sol. $p = 2 > 1$, Hence $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-Series test.

Thm Direct Comparison Test.

If there is an M so that $0 \leq a_n \leq b_n$ for $n \geq M$,

1. If $\sum b_n$ converges then $\sum a_n$ converges
2. If $\sum a_n$ diverges then $\sum b_n$ diverges.

eg | Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$.

Sol | When: $4 \leq n$ $(M=4)$

$2 \leq \frac{n}{2}$

$4 \leq n$

$4n \leq n^2$

$2\sqrt{n} \leq n$

$\sqrt{n} \leq \frac{n}{2}$

$2+\sqrt{n} \leq \frac{n}{2}+\frac{n}{2} = n$

$\frac{1}{n} \leq \frac{1}{2+\sqrt{n}}$

But $\sum \frac{1}{n}$ diverges, so $\sum \frac{1}{2+\sqrt{n}}$ diverges by the direct comparison test.

Q2 | Why does $\sum \frac{1}{n}$ diverge?

Thm

Limit Comparison Test

Let $\{a_n\}$ and $\{b_n\}$ be positive sequences. If

$$L = \lim_{n\to\infty} \frac{a_n}{b_n},$$

then

① If $L > 0$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.

② If $L = \infty$ and $\sum a_n$ converges, then $\sum b_n$ converges.

③ If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

eg | Determine the convergence of $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^4-n-1}$.

Sol | Let $a_n = \dfrac{n^2}{n^4-n-1}$ and $b_n = \dfrac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{n^2}{n^4-n-1}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \left(\frac{n^2}{n^4-n-1}\right)\left(\frac{n^2}{1}\right)$$

$$= \lim_{n\to\infty} \frac{n^4}{n^4-n-1} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{n^3}-\frac{1}{n^4}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 = L > 0.$$

Also $\sum b_n = \sum \frac{1}{n^2}$ converges by the p-series test with $p=2>1$.

Then by the limit comparison test, $\sum a_n = \sum \frac{n^2}{n^4-n-1}$ also converges.

eg | Determine the convergence of $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$.

Sol | Let $a_n = \frac{\sqrt{n}}{n^2+1}$

$b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$.

$\sum b_n = \sum \frac{1}{n^{3/2}}$ converges since $p = \frac{3}{2} > 1$ by the p-series test.

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{\sqrt{n}}{n^2+1}\right)}{\left(\frac{\sqrt{n}}{n^2}\right)} = \lim_{n\to\infty} \left(\frac{\cancel{\sqrt{n}}}{n^2+1}\right)\left(\frac{n^2}{\cancel{\sqrt{n}}}\right)$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1 > 0$$

So by the limit comparison test, $\sum a_n = \sum \frac{\sqrt{n}}{n^2+1}$ converges.