10.5 - Ratio and Root Tests

Thm Ratio Test

Assume the following limit exists:

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

① If $\rho < 1$, then $\sum a_n$ converges absolutely.

② If $\rho > 1$, then $\sum a_n$ diverges.

③ If $\rho = 1$, the test is inconclusive.

Thm Root Test

Assume the limit exists:

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

① If $L < 1$, $\sum a_n$ converges absolutely.

② If $L > 1$, $\sum a_n$ diverges.

③ If $L = 1$, the test is inconclusive.

eg | Determine the convergence of $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$.

Sol. | Try root test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n}{2n+3}\right)^n\right|}$$

$$= \lim_{n\to\infty} \left[\left(\frac{n}{2n+3}\right)^n\right]^{1/n}$$

$$= \lim_{n\to\infty} \frac{n}{2n+3}$$

$$= \frac{1}{2} < 1.$$

So $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$ converges by the root test.

eg | Determine the convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{n^2\, 2^{n+1}}{3^n}$.

Sol | Try ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^2\, 2^{n+2}}{3^{n+1}}}{\frac{n^2\, 2^{n+1}}{3^n}} \right|$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)^2\, 2^{n+2}}{3^{n+1}} \right) \left(\frac{3^n}{n^2\, 2^{n+1}} \right)$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)^2}{n^2} \right) \left(\frac{2^{n+2}}{2^{n+1}} \right) \left(\frac{3^n}{3^{n+1}} \right)$$

$$= \lim_{n\to\infty} \left(\frac{n^2+2n+1}{n^2} \right) \left(\frac{2}{1} \right) \left(\frac{1}{3} \right)$$

$$= \frac{2}{3} \lim_{n\to\infty} \frac{n^2+2n+1}{n^2} = \frac{2}{3}(1) = \frac{2}{3} < 1.$$

Q1] Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{3} < 1$,

the $\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ converges by the ratio test.

T/F.

A] T

eg | Determine the convergence of $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$.

Sol. $\sum_{n=1}^{\infty} \left(\frac{e^2}{n}\right)^n$ Try root test.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{e^{2n}}{n^n}\right|}$$

$$= \lim_{n\to\infty} \left(\frac{e^{2n}}{n^n}\right)^{1/n}$$

$$= \lim_{n\to\infty} \frac{e^{2n/n}}{n^{n/n}}$$

$$= \lim_{n\to\infty} \frac{e^2}{n}$$

$$= 0 < 1.$$

So the $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$ converges by

Q2 | what test?

A2 | root test.

eg | Determine the convergence of $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$.

Sol | no powers, so try ratio test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{\sqrt{n+1}}{n+2}}{\frac{\sqrt{n}}{n+1}} = \lim_{n\to\infty} \left(\frac{\sqrt{n+1}}{n+2}\right)\left(\frac{n+1}{\sqrt{n}}\right).$$

$$= \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} \cdot \left(\frac{n+1}{n+2}\right)$$

$$= (1)(1)$$
$$= 1 \not< 1$$

So the ratio is inconclusive.

Sol

Let $b_n = |a_n| = \dfrac{\sqrt{n}}{n+1}$.

Define $f(x) = \dfrac{\sqrt{x}}{x+1}$, $f(n) = b_n$.

$$f'(x) = \frac{(x+1)\left[\frac{1}{2}x^{-1/2}\right] - \sqrt{x}\,(1)}{(x+1)^2}$$

$$= \frac{(x+1)}{2\sqrt{x}\,(x+1)^2} - \frac{\sqrt{x}}{(x+1)^2}$$

$$= \frac{(x+1) - 2x}{2\sqrt{x}\,(x+1)^2}$$

$$f'(x) = \frac{1-x}{2\sqrt{x}\,(x+1)^2}$$

when $x > 1$ $\quad f'(x) < 0$

$$= \frac{(-)}{(+)}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sqrt{x}}{x+1} \overset{L.H.}{=} \lim_{x\to\infty} \frac{1/2\sqrt{x}}{1} = \lim_{x\to\infty} \frac{1}{2\sqrt{x}}$$

$$= 0.$$

Hence $\{b_n\} \longrightarrow 0$.

By the alternating series test

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1} \quad \text{converges.}$$