10.6 - Power Series

Def A **Power Series** with **center c** is an infinite

Series

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

where x is a variable.

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eg $$F(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

is a power series with center $c=0$.

Thm Radius of Convergence.

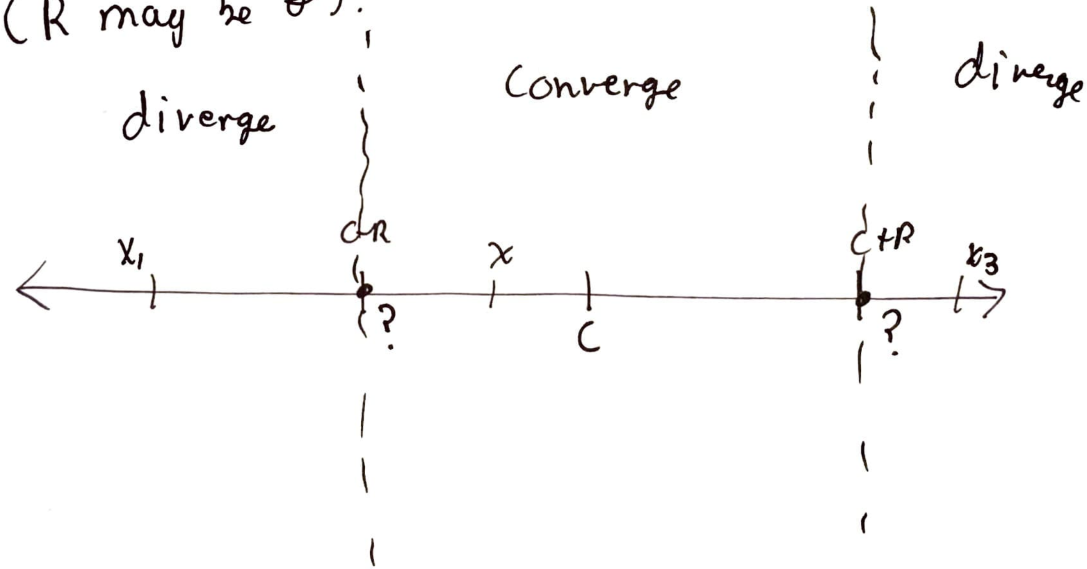
Every power series $F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$,

has a radius of convergence $R \geq 0$, so that

$F(x)$ converges absolutely when $x \in (c-R, c+R)$

and diverges when $x \notin [c-R, c+R]$.

(R may be ∞).



eg | Where does $F(x) = \sum_{n=0}^{\infty} \frac{x^n}{3^n}$ Converge?

Sol. $c = 0$.

Let $a_n = \frac{x^n}{3^n} = \left(\frac{x}{3}\right)^n$

Using root test, $\sum a_n$ converges when

$$1 > L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{x}{3}\right)^n\right|}$$

$$= \lim_{n\to\infty} \left|\left(\frac{x}{3}\right)^n\right|^{1/n} = \lim_{n\to\infty} \left|\left(\frac{x}{3}\right)^{n/n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x}{3}\right|$$

$1 > L = \left|\frac{x}{3}\right|$

$3 > |x|$ when $|x| < 3$

Sol

Check end points:

$x=3$

$$F(3)=\sum_{n=0}^{\infty}\frac{3^n}{3^n}=\sum_{n=0}^{\infty}1 \quad \text{diverges by div. test}$$

$$F(-3)=\sum_{n=0}^{\infty}\frac{(-3)^n}{3^n}=\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{3^n}$$

$$=\sum_{n=0}^{\infty}(-1)^n=1-1+1-1\cdots$$

diverges.

$F(x)$ converges for $x\in(-3,3)$.

Thm Let $F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$, with radius of convergence $R > 0$. Then

① $F'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$

② $\int F(x)\,dx = K + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-c)^{n+1}$

for K a constant of integration.

① and ② have radius of convergence R.

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pf Idea. ① $F(x) = a_0 + a_1 (x-c)^1 + a_2 (x-c)^2 + a_3 (x-c)^3 + \cdots = \sum_{n=0}^{\infty} a_n (x-c)^n$

$F'(x) = 0 + a_1 + 2a_2 (x-c) + 3a_3 (x-c)^2 + \cdots$

$F'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$

Q1| Verify #2 in a similar way. Answer T when you have done so.

eg Find $F'(x)$ and $\int F(x)\,dx$ for

$$F(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \underbrace{1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots}_{} = e^x$$

Sol. Recall: $n! = n\cdot(n-1)(n-2)\cdots(1)$, with $0! = 1$.

$$F'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$F'(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = F(x).$$

Thus $F(x) = e^x$.

$$\int F(x)\,dx = K + F(x) = K + \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Thm $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, for $|x| < 1$.

Pf $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$

is a geometric series with first term 1 and ratio $r = x$. This converges when $|x| < 1$ to

$$\frac{1}{1-x}.$$

eg | Find a power series for $\frac{1}{(1-x)^2}$

Sol. $\left(\frac{1}{1-x}\right)' = \left[(1-x)^{-1}\right]' = -1(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ So

$\frac{1}{(1-x)^2} = \left[\sum_{n=0}^{\infty} x^n\right]' = \underline{\sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots}$

eg | Find a power series representation and radius of convergence for $F(x) = \frac{1}{1-ax}$, where $a>0$.

Sol | $F(x) = \frac{1}{1-ax}$. Let $y = ax$

$$= \frac{1}{1-y} = \sum_{n=0}^{\infty} y^n = \sum_{n=0}^{\infty} (ax)^n = \sum_{n=0}^{\infty} a^n x^n.$$

The radius of convergence is

$1 > |y| = |ax| = a|x|$.

So $|x| < \frac{1}{a}$, radius of conv. is $\frac{1}{a}$.

Take $a=2$. This is example ____ in the book.

Q2| What goes here?

Con Skip D, E. part pg 558-560.