10.7 - Taylor Series

Thm

Taylor Series Expansion

If $f(x)$ is represented by a power series centered at c, in an interval $(c-R, c+R)$ for $R>0$, then that power series is the Taylor Series

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

When $c=0$, $T(x)$ is called the Maclaurin Series.

$$M(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Pf

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots a_n(x-c)^n + \cdots$$

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots + na_n(x-c)^{n-1} + \cdots$$

$$f''(x) = 2a_2 + 3(2)a_3(x-c) + \cdots + n(n-1)a_n(x-c)^{n-2} + \cdots$$

$\vdots$

$$f^{(n)}(x) = n(n-1)(n-2)\cdots\underbrace{(n-(n-1))}_{1} a_n + \cdots$$

$\vdots$

$f(c) = a_0$

$f'(c) = a_1$

$f''(c) = 2a_2$

$\vdots$

$f^{(n)}(c) = n!\,a_n$

$$a_n = \frac{f^{(n)}(c)}{n!}$$

eg | Let $f(x) = x^3$. Find a Taylor Series Expansion for f centered at $x = 2$.

Sol.

$f(x) = x^3$ $\qquad$ $f(2) = 8$

$f'(x) = 3x^2$ $\qquad$ $f'(2) = 12$

$f''(x) = 6x$ $\qquad$ $f''(2) = 12$

$f^{(3)}(x) = 6$ $\qquad$ $f^{(3)}(2) = 6$

$f^{(4)}(x) = 0$ $\qquad$ $f^{(n)}(2) = 0$ for $n \geq 4$.

$\vdots$

$f^{(n)}(x) = 0$ for $n \geq 4$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$$= \frac{f(c)}{0!}(x-c)^0 + \frac{f'(c)}{1!}(x-c)^1 + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + 0$$

$$= \frac{8}{1}(1) + \frac{12}{1}(x-c) + \frac{12}{2!}(x-c)^2 + \frac{6}{3!}(x-c)^3 + 0$$

$$f(x) = 8 + 12(x-2) + 6(x-2)^2 + (x-2)^3$$

Thm 2 in the Book.

↳ Read it, understand it,

eg Find the Maclaurin Series for cos(x).

Sol.

$f^{(0)}(x) = \cos x$ $\quad f^{(0)}(0) = 1$ $\quad 1, 0, -1, 0, 1, 0, -1, 0 \ldots$

$f^{(1)}(x) = -\sin x$ $\quad f^{(1)}(0) = 0$

$f^{(2)}(x) = -\cos x$ $\quad f^{(2)}(0) = -1$

$f^{(3)}(x) = \sin x$ $\quad f^{(3)}(0) = 0$

$f^{(4)}(x) = \cos x$ $\quad f^{(4)}(0) = 1$

$\vdots \qquad \vdots$

So $\cos x = f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$$= \frac{f^{0}(0)}{1} x^0 + \frac{f^{(1)}(0)}{1} x^1 + \frac{f^{(2)}(0)}{2} \cdot x^2 + \cdots$$

$$= (1) + 0x^1 - \frac{x^2}{2} + 0x^3 + \frac{x^4}{4!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

Thm ① $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

② $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Pf ① Done. ✓

② Q11 Write this out by hand.
4-non-zero terms. Answer T
when complete.

Thm

Operations on Power Series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$

$g(x) = \sum_{n=0}^{\infty} b_n x^n$.

① $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$.

② $f(x^N) = \sum_{n=0}^{\infty} a_n x^{nN}$

③ $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$.

Note

These operations can change the radius of convergence. For example in ③ the radius of convergense is the intersection of the ~~radii~~ radii of convergence.

eg | Find the Maclaurin Series for $f(x) = \frac{3x-1}{x^2-1}$ and find its radius of convergence.

Sol | $\frac{3x-1}{x^2-1} = \frac{3x-1}{(x-1)(x+1)} = \frac{A_1}{x-1} + \frac{A_2}{x+1}$

$3x-1 = A_1(x+1) + A_2(x-1)$ eval at $x=1$

$3(1)-1 = A_1(2) + A_2 0$

$2 = 2A_1$

$A_1 = 1$

$3x-1 = (x+1) + A_2(x-1)$ eval at $x=-1$

Q2| Find A_2.

$3(-1)-1 = 0 + A_2(-2)$

$-4 = -2A_2$

$2 = A_2$

Sol

$$f(x) = \frac{3x-1}{x^2-1} = \frac{2}{x+1} + \frac{1}{x-1}$$

$$\frac{1}{x-1} = \frac{1}{(-1)(1-x)} = \frac{-1}{1-x} = -1\left(\frac{1}{1-x}\right)$$

$$\frac{1}{x-1} = -1 \sum_{n=0}^{\infty} x^n, \quad \text{radius of conv. } R=1.$$

$$\frac{2}{x+1} = 2\left(\frac{1}{1-(-x)}\right) = 2\sum_{n=0}^{\infty} (-x)^n$$

$$\frac{2}{x+1} = \sum_{n=0}^{\infty} 2(-1)^n x^n$$

$$f(x) = \sum_{n=0}^{\infty} 2(-1)^n x^n + \sum_{n=0}^{\infty} (-1) x^n$$

$$= \sum_{n=0}^{\infty} 2(-1)^n x^n + (-1) x^n$$

$$f(x) = \sum_{n=0}^{\infty} \left[2(-1)^n - 1 \right] x^n$$

$$f(x) = 1 - 3x + 1x^2 - 3x^3 + x^4 + \cdots$$

So $R = 1$, i.e. valid for $x \in (-1, 1)$.

eg | Find the maclaurin Series for $x^4 \cos x$.

Sol | $x^4 \cos x = x^4 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \right]$

$$x^4 \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+4}}{(2n)!}$$

Def The <u>binomial coefficients</u> for $a \in \mathbb{R}$ and $0 \leq n$ an integer is

$$\binom{a}{n} = \frac{a(a-1)(a-2)\cdots(a-(n-1))}{n(n-1)(n-2)\cdots 1} = \frac{a(a-1)\cdots(a-(n-1))}{n!}$$

and $\binom{0}{0} = 1$.

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eg $$\binom{7}{3} = \frac{7(7-1)(7-2)}{3\cdot 2\cdot 1} = \frac{7(\cancel{6})(5)}{\cancel{6}} = 35.$$

$$\binom{1/2}{3} = \frac{(1/2)(1/2-1)(1/2-2)}{3\cdot 2\cdot 1} = \frac{(1/2)(-1/2)(-3/2)}{6}$$

$$= \frac{3/8}{6} = \frac{\cancel{3}}{8\cdot\cancel{3}\cdot 2} = \frac{1}{16}$$

$$\binom{1}{0} = \frac{1}{0!} = \frac{1}{1} = 1$$

$$\binom{n}{n} = \frac{n(n-1)(n-2)\cdots(n-(n-1))}{n!} = 1.$$

Thm | Binomial Theorem

For $a \in \mathbb{R}$

$$(x+y)^a = \sum_{k=0}^{a} \binom{a}{k} x^{a-k} y^k$$

eg

$$(x+y)^0 = \binom{0}{0} x^0 y^0 = \underline{1}$$

$$(x+y)^1 = \binom{1}{0} x^{1-0} y^0 + \binom{1}{1} x^{1-1} y^1 = \underline{1}x + \underline{1}y$$

$$(x+y)^2 = \binom{2}{0} x^{2-0} y^0 + \binom{2}{1} x^{2-1} y^1 + \binom{2}{2} x^{2-2} y^2 = \underline{1}x^2 + \underline{2}xy + \underline{1}y^2$$

$$(x+y)^3 = \binom{3}{0} x^3 y^0 + \binom{3}{1} x^2 y + \binom{3}{2} x y^2 + \binom{3}{3} y^3 = x^3 + \underline{3}x^2y + \underline{3}xy^2 + y^3$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Thm

Binomial Series

For any exponent a and $|x| < 1$,

$$(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n.$$