11.1 - Parametric Equations

Def If f and g are continuous functions of t on an interval I, then

$$x = f(t) \text{ and } y = g(t)$$

are called <u>parametric equations</u>, and t is called the <u>parameter</u>. The set of points (x, y) obtained ~~by~~ as t varies over I is called the <u>graph</u> of the parametric equations.

$c(t) = (f(t), g(t))$ is called a <u>parametrization</u>.

eg | Sketch $x(t) = 2t - 4$

$y(t) = 3 + t^2$

Sol | $x(t)$ and $y(t)$ are the parametric equations.

$c(t) = (x(t), y(t))$ is a parameterization, with parameter t.

Eliminate the parameter t:

$$x = 2t - 4$$

$$\frac{x-4}{2} = t$$

$$t = \frac{x}{2} - 2$$

$$y = 3 + t^2 = 3 + \left(\frac{x}{2} - 2\right)^2$$

$$y = 7 + 2x + \frac{1}{4}x^2.$$

Thm Parameterization of a line

① The line thru $P = (a,b)$ of slope m is parametrized by

$$x(t) = a + rt$$
$$y(t) = b + st$$

for $t \in \mathbb{R}$, $r \neq 0$, and $m = \frac{s}{r}$.

② The line thru $P = (a,b)$ and $Q = (c,d)$ is parameterized by

$$x(t) = a + t(c-a)$$
$$y(t) = b + t(d-b).$$

for $t \in \mathbb{R}$.

$c(t) = (x(t), y(t))$; $\quad c(0) = P$
$\qquad c(1) = Q$.

eg | Find a parameterization for the line between

$P = (3,1)$ and $Q = (7,13)$.

Sol | $\ell(t) = (x, y)$

$= (a + t(c-a),\ b + t(d-b))$

$= (3 + t(7-3),\ 1 + t(13-1))$

$\ell(t) = (3 + 4t,\ 1 + 12t)$

$\ell(0) = (3, 1) = P$

$\ell(1) = (7, 13) = Q$.

Des mos.

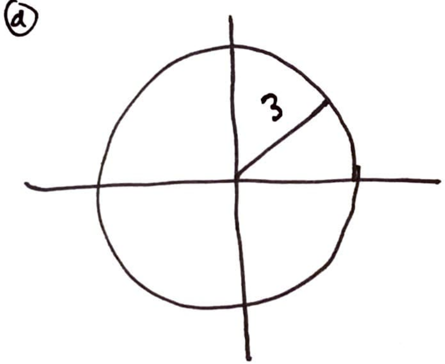
Rem: It is easy to translate/scale when using parametric equations.

eg: Consider ⓐ $(3\cos t, 3\sin t)$ and

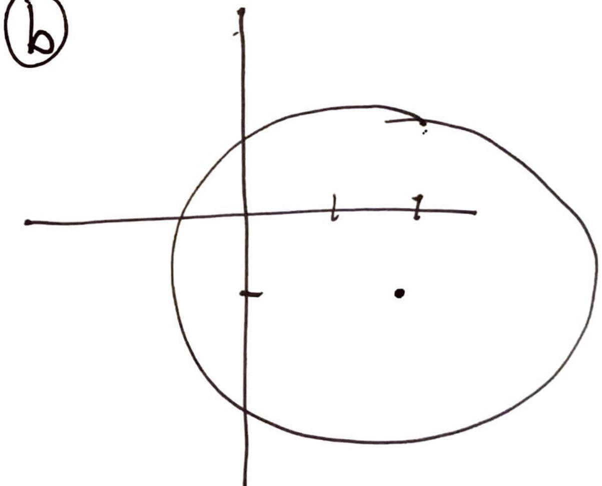
$(2+3\cos t, -1+3\sin t)$

for $t \in [0, 2\pi]$.

ⓐ



ⓑ



Desmos.

Thm Let $c(t) = (x(t), y(t))$, where x' and y' exist and x' is continuous and $x'(t) \neq 0$.

Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}.$$

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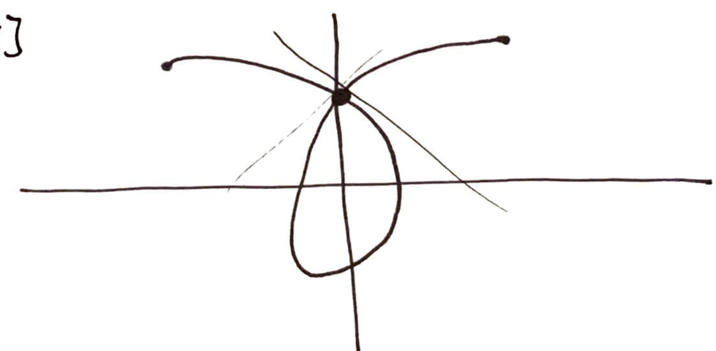
eg Find the eqns. of tangent lines at $t = -\frac{\pi}{2}$ and $t = \frac{\pi}{2}$ for

$$c(t) = (2t - \pi \sin t, 2 - \pi \cos t)$$

Sol $t \in [-\pi, \pi]$



Sol

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\pi \sin t}{2 - \pi \cos t}$$

$$\left.\frac{dy}{dx}\right|_{t=-\frac{\pi}{2}} = \frac{\pi \sin(-\frac{\pi}{2})}{2-\pi \cos(-\frac{\pi}{2})} = \frac{\pi(-1)}{2-\pi(0)} = -\frac{\pi}{2}$$

$$\left.\frac{dy}{dx}\right|_{t=-\frac{\pi}{2}} = -\frac{\pi}{2}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = \frac{\pi}{2}$$

$$C(-\tfrac{\pi}{2}) = \left(2(-\tfrac{\pi}{2}) - \pi\sin(-\tfrac{\pi}{2}),\ 2-\pi\cos(-\tfrac{\pi}{2})\right)$$
$$= (-\pi + \pi,\ 2-0)$$

$$C(-\tfrac{\pi}{2}) = (0, 2)$$
$$C(\tfrac{\pi}{2}) = (0, 2)$$

Sol

$$y-2 = -\frac{\pi}{2}x$$

$$y-2 = \frac{\pi}{2}x$$

Thm The area under a parametric curve $c(t) = (x(t), y(t))$ which is always above the x-axis, and traces the graph of a function is

$$A = \int_{t_0}^{t_1} y(t)\, x'(t)\, dt.$$