11.2 - Arc Length and Speed

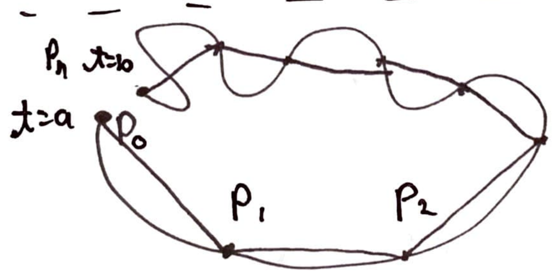
Thm

Arc length

Let $c(t) = (x(t), y(t))$, where x' and y' exist and are continuous. Then the arc length s of $c(t)$ for $a \le t \le b$ is

$$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$$

Pf.



Create a partition of size n of $[a, b]$,

$P = a = t_0 < t_1 < t_2 \cdots < t_n = b$

Define $P_n = c(t_n)$

$P_n = (x(t_n), y(t_n))$

Pf $d(P_{i-1}, P) = d\big((x(t_{i-1}), y(t_{i-1})), (x(t_i), y(t_i))\big)$

$= $ Q1] write out this formula,

Select True when finished.

A1]

$= \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2}$

By the MVT there are

$c_{x_i}, c_{y_i} \in [t_{i-1}, t_i]$ so that

$$x'(c_{x_i}) = \frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}}$$ or

$x'(c_{x_i}) \Delta t_i = x(t_i) - x(t_{i-1})$. Similar for y!

$y'(c_{y_i}) \Delta t_i = y(t_i) - y(t_{i-1})$.

Pf | So $d(P_{i-1}, P_i) = \sqrt{[x'(c_{x_i})\Delta t_i]^2 + [y'(c_{y_i})\Delta t_i]^2}$

$$d(P_{i-1}, P_i) = \sqrt{[x'(c_{x_i})]^2 + [y'(c_{y_i})]^2}\ \Delta t_i$$

$$\underline{s} = \lim_{N\to\infty} \sum_{i=1}^{N} d(P_{i-1}, P_i) = \underline{\int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt}$$

Rem. For a function $f(x)$, a parametrization for $f(x)$ is $c(t) = (t, f(t))$. Then

$$s = \int_a^b \sqrt{1 + [f'(t)]^2}\, dt$$

eg | Calculate the arc length of a circle of radius R.

Sol | $c(t) = (R\cos t, R\sin t)$ for $t \in [0, 2\pi]$.

$x' = -R\sin t$

$y' = R\cos t$

$$S = \int_a^b \sqrt{x'^2 + y'^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(-R\sin t)^2 + (R\cos t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{R^2\sin^2 t + R^2\cos^2 t}\, dt$$

$$= \int_0^{2\pi} R\sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= R\int_0^{2\pi} dt$$

$$S = R[2\pi - 0] = 2\pi R.$$

Thm | Speed along a parametrization

The speed of $c(t) = (x(t), y(t))$ is

$$\text{Speed} = \frac{ds}{dt} = \frac{d}{dt}\int \sqrt{x'^2 + y'^2}\, dt$$

$$\text{Speed} = \sqrt{x'^2 + y'^2}$$

eg | A particle travels along the path $c(t) = (2t, 1+t^{3/2})$. Find the speed at $t=1$.

Sol | $\text{Speed} = \sqrt{2^2 + \left(\frac{3}{2}t^{1/2}\right)^2} = \sqrt{4 + \frac{9}{4}t}$

$\text{Speed}\big|_{t=1} = \sqrt{4 + \frac{9}{4}}$. Q2| Simplify to an irreducible fraction.

$$= \sqrt{\frac{16+9}{4}} = \frac{\sqrt{25}}{\sqrt{4}} = \frac{5}{2}.$$

Thm

Area of a Surface of Revolution

Suppose $c(t) = (x(t), y(t))$ is smooth (x' and y' exist) and does not cross itself on $[a, b]$. The the surface area S of the surface of revolution formed by revolving c about a coord. axis is

① $S = 2\pi \int_a^b y(t) \sqrt{x'^2 + y'^2}\, dt$

about the x-axis, $y(t) \geq 0$.

② $S = 2\pi \int_a^b x(t) \sqrt{x'^2 + y'^2}\, dt$

about the y-axis, $x(t) \geq 0$.

Remark

$$s = \int \sqrt{x'^2 + y'^2}\, dt$$

$$\frac{ds}{dt} = \sqrt{x'^2 + y'^2}$$

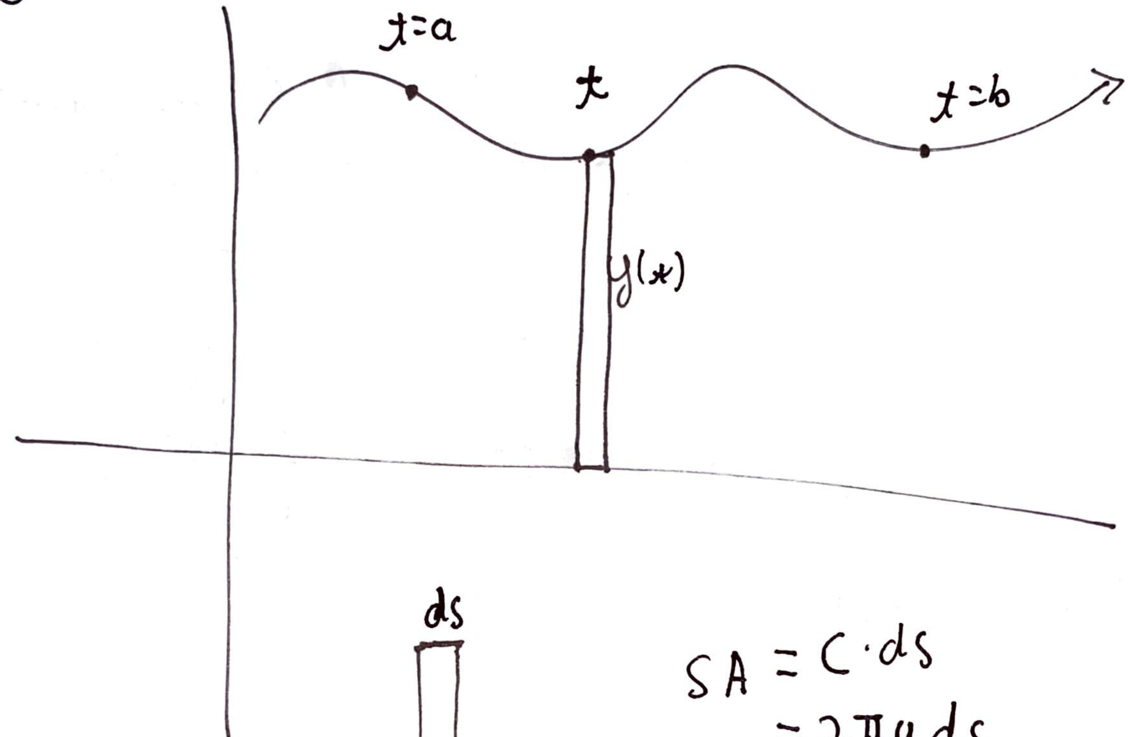
$$ds = \sqrt{x'^2 + y'^2}\, dt$$

① $S = 2\pi \int_a^b y \, ds$

② $S = 2\pi \int_a^b x \, ds$.

Rem

①



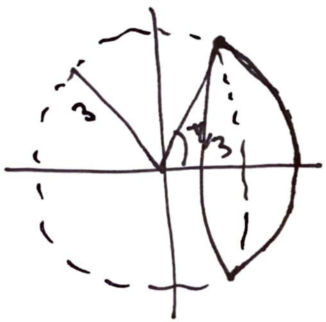
$$SA = C \cdot ds$$
$$= 2\pi y \, ds$$

$$SA = \int_a^b 2\pi y \, ds$$

eg | Let $C(t) = (3\cos t, 3\sin t)$ for $t \in [0, \cancel{2\pi}\ \pi/3]$.

Find the S.A. of the surface bound by revolving C about the x-axis.

Sol



$$S = 2\pi \int_a^b y \, ds$$

$$= 2\pi \int_a^b y \sqrt{x'^2 + y'^2} \, dt$$

$$= 2\pi \int_0^{\pi/3} 3\sin t \sqrt{(-3\sin t)^2 + (3\cos t)^2} \, dt$$

$$= 18\pi \int_0^{\pi/3} \sin t \, dt$$

$$= 18\pi \left[-\cos t\right]_0^{\pi/3}$$

$$= -18\pi \left[\cos \pi/3 - \cos 0\right]$$

$$= -18\pi \left[1/2 - 1\right]$$

$$S = 9\pi$$