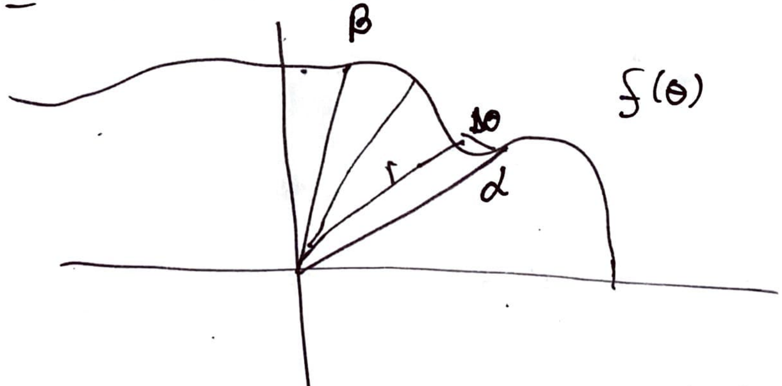
11.4 - Arc Length and Area in Polar Coordinates

Thm If f is a continuous function, the the area bounded by a curve in polar form $r = f(\theta)$, and the rays $\theta = \alpha$ and $\theta = \beta$, with $\alpha < \beta$

is

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2 \, d\theta .$$

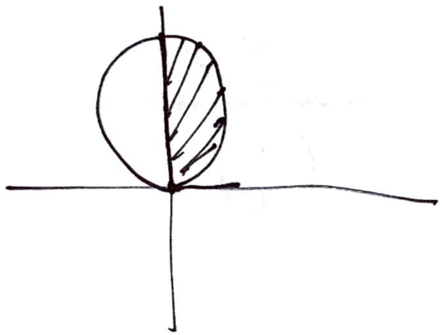
pf.



Use area of a Sector, $A = \frac{1}{2} r^2 \Delta\theta$

eg: Use the previous theorem to compute the area of the right semicircle of radius 2, tangent to the x-axis at the origin.

Sol



$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi/2} (4\sin\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi/2} 16\sin^2\theta\,d\theta$$

$$= 8\int_{0}^{\pi/2} \frac{1}{2}(1-\cos 2\theta)\,d\theta$$

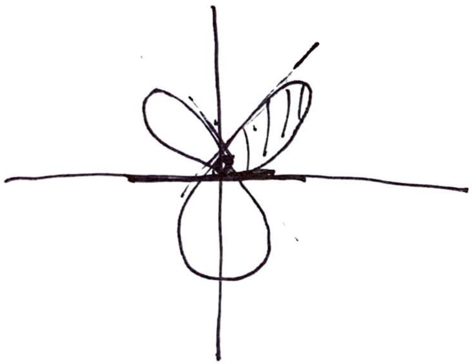
$$= 4\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= 4\left[\pi/2 - 0\right] - 4\left[0 - \frac{0}{2}\right]$$

$$= 2\pi$$

eg | Sketch $r = \sin(3\theta)$ and compute the area of one petal.

Sol.



$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi/3} \sin^2(3\theta) \, d\theta$$

$$\vdots$$

$$= \frac{\pi}{12}$$

Def | The region between two polar curves $r = f_1(\theta)$ and $r = f_2(\theta)$ with $f_2(\theta) \geq f_1(\theta) \geq 0$ for $\alpha \leq \theta \leq \beta$, is radially simple.

Thm The area between two radially simple continuous curves in polar form is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} (f_2(\theta))^2 - (f_1(\theta))^2 \, d\theta .$$

Thm Arc length $= \int_{\alpha}^{\beta} \sqrt{(f(\theta))^2 + (f'(\theta))^2} \, d\theta$

Pf. Q1| Work this out. See Book, after example 6, up to equation 7.

Rem This is <u>used</u> and is <u>useful</u> in Calc III.