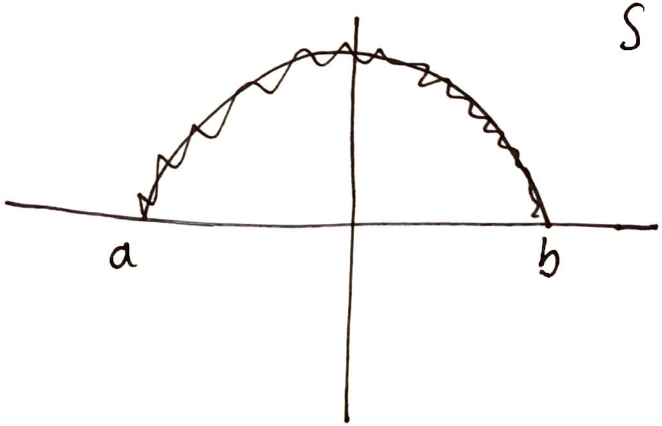
8.1 - Arc Length and Surface Area

Thm Let f be a function so that f' is continuous on $[a,b]$, Then the arc length s of $y=f(x)$ over $[a,b]$ is

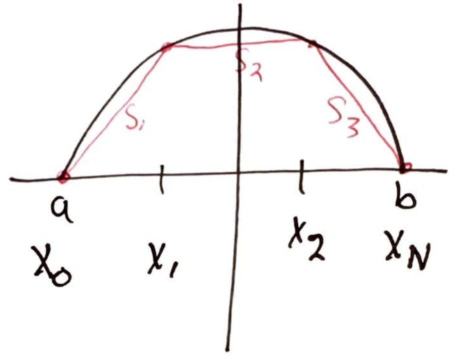
$$s = \int_a^b \sqrt{1+[f'(x)]^2}\, dx.$$

Pf



s = length of the curve.

Pf



$$S \approx \sum_{i=1}^{N} |S_i| = \sum_{i=1}^{N} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

$$= \sum_{i=1}^{N} (x_i - x_{i-1}) \sqrt{1 + \frac{(f(x_i) - f(x_{i-1}))^2}{(x_i - x_{i-1})^2}}$$

$$= \sum_{i=1}^{N} \sqrt{1 + \frac{(f(x_i) - f(x_{i-1}))^2}{(x_i - x_{i-1})^2}} \, \Delta x$$

By MVT $\exists\, c_i \in (x_{i-1}, x_i)$ s.t.

$f'(c_i) = [f(x_i) - f(x_{i-1})] / [x_i - x_{i-1}]$.

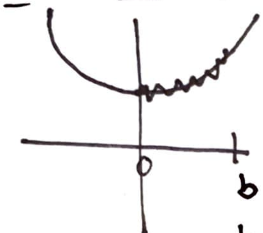
$$S \approx \sum_{i=1}^{N} \sqrt{1+[f'(c_i)]^2}\, \Delta x$$

$$S = \lim_{N \to \infty} \sum_{i=1}^{N} \sqrt{1+[f'(c_i)]^2}\, \Delta x$$

$$S = \int_a^b \sqrt{1+[f'(x)]^2}\, dx .$$

eg Find the arc length $S(b)$ of $y = \cosh(x)$ over $[0, b]$.

Sol



$$S(b) = \int_0^b \sqrt{1 + [(\cosh x)']^2}\, dx$$

$$= \int_0^b \sqrt{1 + [\sinh x]^2}\, dx$$

$$= \int_0^b \sqrt{\cosh^2 x}\, dx$$

$$= \int_0^b \cosh x\, dx$$

$$= \sinh x \Big|_0^b$$

$$= \sinh b - \sinh 0$$

$$S(b) = \sinh b$$

eg | Find the arc length of the graph of

$$f(x) = \frac{x^3}{6} + \frac{1}{2x} \text{ on } [1/2, 2].$$

Sol

$$S = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \qquad \text{Arc leng. thm.}$$

$$= \int_{1/2}^{2} \sqrt{1 + \left[\left(\frac{x^3}{6} + \frac{1}{2x}\right)'\right]^2}\, dx \qquad \text{Sub.}$$

$$= \int_{1/2}^{2} \sqrt{1 + \left[\frac{3x^2}{6} + \frac{1}{2}x^{-2}(-1)\right]^2}\, dx \qquad \text{der. of } f.$$

$$= \int_{1/2}^{2} \sqrt{1 + \left[\frac{x^2}{2} - \frac{1}{2x^2}\right]^2}\, dx \qquad \text{Simp.}$$

$$= \int_{1/2}^{2} \sqrt{1 + \frac{x^4}{4} - 2\left(\frac{x^2}{2}\right)\left(\frac{1}{2x^2}\right) + \frac{1}{4x^4}}\, dx \qquad \text{expand}$$

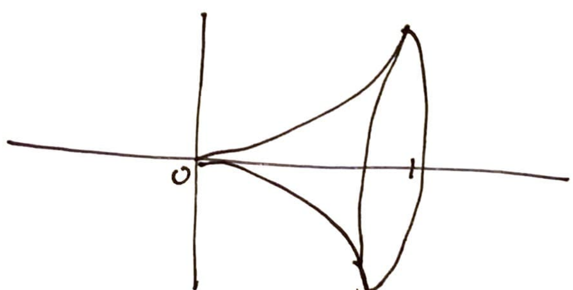
Sol	$= \int_{1/2}^{2} \sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}}\, dx$	Simp.
	$= \int_{1/2}^{2} \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}}\, dx$	
	$= \int_{1/2}^{2} \frac{1}{2} \sqrt{x^4 + 2 + \frac{1}{x^4}}\, dx$	
	$= \frac{1}{2} \int_{1/2}^{2} \sqrt{(x^2 + \frac{1}{x^2})^2}\, dx$	
	$= \frac{1}{2} \int_{1/2}^{2} (x^2 + \frac{1}{x^2})\, dx$	
	$= \frac{1}{2} \left[\frac{x^3}{3} - x^{-1}\right]_{1/2}^{2}$	FTC
	$= \frac{33}{16}$	Ans

thm Let $f(x) \geq 0$ and let $f'(x)$ be continuous on $[a,b]$. The Surface area S of the surface obtained by rotating the graph of f about the x-axis for $a \leq x \leq b$ is

$$S = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\, dx$$

eg Find the surface formed by revolving $f(x) = x^3$ on $[0,1]$ about the x-axis.

Sol



Sol

$$S = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$$ S.A. thm

$$= 2\pi \int_0^1 x^3 \sqrt{1+[(x^3)']^2}\,dx$$ Sub.

$$= 2\pi \int_0^1 x^3 \sqrt{1+[3x^2]^2}\,dx$$ dir. expand

$$= 2\pi \int_0^1 x^3 \sqrt{1+9x^4}\,dx$$ simp.

Let $u = 1+9x^4$

$\frac{du}{dx} = 36x^3$

$\frac{du}{36x^3} = dx$

$$= 2\pi \int_0^1 \cancel{x^3} \sqrt{u}\,\frac{du}{36\cancel{x^3}}$$

$$= \frac{\pi}{18}\int_{0=x}^{1=x} u^{1/2}\,du$$

$$= \frac{\pi}{18} u^{3/2}\left(\frac{2}{3}\right)\Big]_1^{10}$$

$$S = \frac{\pi}{27}\left[10^{3/2} - 1\right] \approx 3.56$$