$z = a + bi$, $\bar{z} = a - bi$

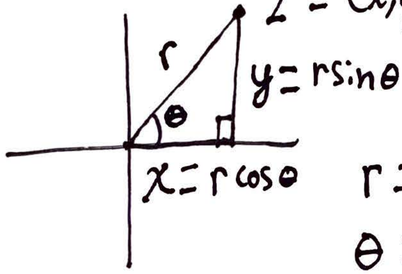
$$z \cdot \bar{z} = (a+bi)(a-bi)$$
$$= a^2 \cancel{-abi} \cancel{+abi} - b^2 i^2$$
$$z \cdot \bar{z} = a^2 + b^2$$
$$|z| = \sqrt{a^2+b^2} = \sqrt{z\bar{z}}$$

Thm: The quotient of $z = a+bi$ and $\overset{\neq 0}{w} = c+di$ is

$$\frac{z}{w} = \left(\frac{a+bi}{c+di}\right)\left(\frac{c-di}{c-di}\right) = \frac{ac - adi + bci - bdi^2}{c^2+d^2}$$
$$= \frac{ac + db + (bc - ad)i}{c^2+d^2}$$
$$\frac{z}{w} = \frac{ac+bd}{c^2+d^2} + \left(\frac{bc-ad}{c^2+d^2}\right)i$$

# Polar form of complex numbers

$z = (x, y) = x + iy$




$r = |z|$

$\theta$ = angle btw. + hor. axis and the ray through the point $(x, y)$.

$$\cos\theta = \frac{x}{r}$$

$$\theta = \cos^{-1}\left(\frac{x}{r}\right)$$

Def: The <u>polar form</u> of $z = x + yi$ is

$$z = r\cos\theta + ir\sin\theta$$
$$z = r\,\text{cis}\,\theta.$$

# Complex Exponential

$$\frac{d\, e^{iy}}{d(iy)} = e^{iy}$$

$$\frac{de^{iy}}{d(y)} = \frac{de^{iy}}{d(iy)} \cdot \frac{diy}{dy} = e^{iy} i = i e^{iy}.$$

$$\frac{de^{iy}}{dy} = i e^{iy}$$

$$\frac{d^2 e^{iy}}{dy^2} = i^2 e^{iy} = -e^{iy}$$

$g(y) = e^{iy}$, $\quad g'' = -g \ (*)$

$$(A\cos\theta + B\sin\theta)'' = (-A\sin\theta + B\cos\theta)'$$
$$= -A\cos\theta - B\sin\theta$$

So $A\cos\theta + B\sin\theta$ satisfy $(*)$

$$e^{iy} = g(y) = A\cos\theta + B\sin\theta$$

$e^{iy} = g(y) = A\cos\cancel{\theta}y + B\sin\cancel{\theta}y$

$1 = e^{i\cdot 0} = g(0) = A\cos 0 + B\sin 0 = A + 0$

$\underline{1 = A.}$

$ie^{iy} = g'(y) = \cancel{\cos} (\cos y + B\sin y)'$

$ie^{iy} = g'(y) = -\sin y + B\cos y$

$i = ie^{0} = g'(0) = -\sin 0 + B\cos(0) = B$

$\underline{i = B}$

$e^{iy} = g(y) = \cos y + \cancel{i\sin\theta}\, i\sin y$

$e^{iy} = \cos y + i\sin y$

Euler's formula.

Thm: For $z = x+iy$, $\underline{e^{z} = e^{x+iy} = e^{x}e^{iy} = e^{x}[\cos y + i\sin y].}$

Def: ~~The Alternate form of the~~

The Alternate polar form of $z = r\cos\theta + ir\sin\theta$
$= r\,\mathrm{cis}\,\theta$
is

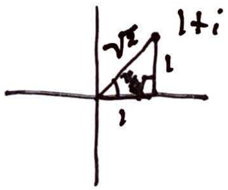
$$z = re^{i\theta}.$$

Fact: Laws of exponents hold for complex exponentiation.

Def: For $z = re^{i\theta}$, the <u>argument</u> of $z$ is

$$\arg(z) = \theta \mod (2\pi).$$

eg: What is $\arg(1+i)$?




$$\arg(1+i) = \frac{\pi}{4}$$

For $z = |z|e^{i\theta}$ and $w = |w|e^{i\beta}$

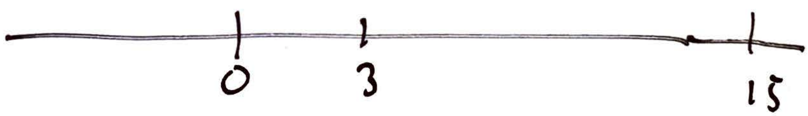
$\underline{z = re^{i\theta}}$ and $\underline{w = se^{i\beta}}$

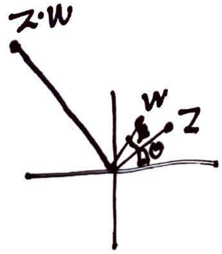
$$z \cdot w = (re^{i\theta})(se^{i\beta})$$
$$= (rs)e^{i\theta}e^{i\beta}$$
$$= (rs)e^{i\theta + i\beta}$$
$$z \cdot w = (rs)e^{i(\theta+\beta)}$$

---

Real multiplication

$3 \cdot 5$



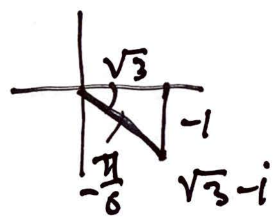



$$\frac{z}{w} = \frac{re^{i\theta}}{se^{i\beta}} = \left(\frac{r}{s}\right) e^{i(\theta-\beta)}$$

eg| Find $\dfrac{1+i}{\sqrt{3}-i}$.

$$1+i = \sqrt{2}\, e^{i\pi/4}$$

$$\sqrt{3}-i = 2e^{i(-\pi/6)}$$




$$r = \sqrt{3+1} = \sqrt{4} = 2$$

$$\frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}\, e^{i\pi/4}}{2\, e^{i(-\pi/6)}}$$

$$= \frac{\sqrt{2}}{2} e^{i(\pi/4 + \pi/6)}$$

$$\frac{i+1}{\sqrt{3}-i} = \frac{\sqrt{2}}{2} e^{i(5\pi/12)}$$

eg] $(1+i)^{24} = \left(\sqrt{2}\, e^{i\pi/4}\right)^{24}$

$= \left(2^{1/2}\right)^{24} \cdot \left(e^{i\pi/4}\right)^{24}$

$= 2^{12} \cdot e^{i\,24\pi/4}$

$= 2^{12} \cdot e^{i6\pi}$

$= 2^{12} \cdot \operatorname{cis} 6\pi$

$= 2^{12} \cdot (1 + i(0))$

$(1+i)^{24} = 2^{12}$

Thm: De Moivre's Theorem.

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

pf: $(\cos\theta + i\sin\theta)^n = \left(e^{i\theta}\right)^n = e^{i(n\theta)} = \cos(n\theta) + i\sin(n\theta).$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$e^{iy} = 1 + (iy) + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \cdots$$

$$= 1 + (iy) + \frac{i^2y^2}{2!} + \frac{i^3y^3}{3!} + \frac{i^4y^4}{4!} + \cdots$$

$$= 1 + iy + \frac{-y^2}{2!} \quad \frac{-iy^3}{3!} + \frac{y^4}{4!} + \cdots$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} \cdots\right)$$

$$e^{iy} = \cos y + i \sin y$$