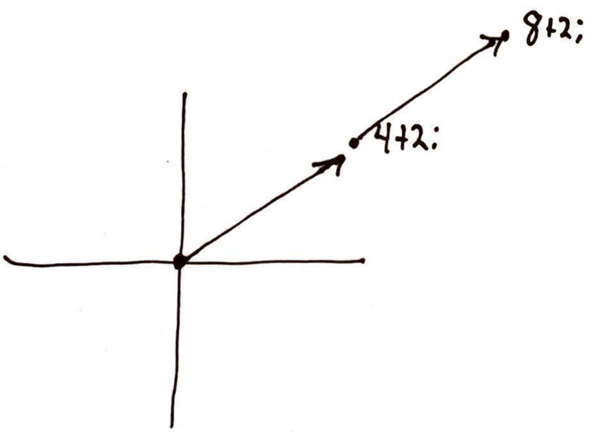
Def: A transformation is a function that is 1-1 and whose Domain is equal to it's codomain.

Rem: We will look at geometric (complex) transformations.

$$T : \mathbb{C} \to \mathbb{C}$$

eg) $f(z) = z + (4+2i)$  $\quad f : \mathbb{C} \to \mathbb{C}$

$$z \mapsto z + (4+2i)$$




Def: A geometric transformation is a translation if it is of the form

$z \mapsto z + w$ for some $w \in \mathbb{C}$.

Def: A geometric transformation is a rotation if it is of the form

$$z \mapsto e^{i\theta} \cdot z,$$

for some $\theta \in \mathbb{R}$.

Recall: $e^{i\theta} = \cos\theta + i\sin\theta$

$z = re^{i\alpha}$, rotate by $\theta$, $z \mapsto e^{i\theta} z$

$$e^{i\theta} z = e^{i\theta} r e^{i\alpha} = r e^{i(\theta+\alpha)}$$

Def: A geometric transformation is homothetic if it is of the form $z \mapsto kz$ for some $0 < k \in \mathbb{R}$. If $k < 1$, then this is a shrinking. If $k > 1$, this is a stretching.

**eg** Classify each of the following as translation, rotation or homothetic:

ⓐ $z \mapsto z+2$

ⓑ $z \mapsto 5z$

ⓒ $z \mapsto z+i-3$

ⓓ $z \mapsto iz$

ⓐ $z \mapsto z+2 = z+\underset{\substack{\uparrow \\ \mathbb{C}}}{(2+0i)}$ translation.

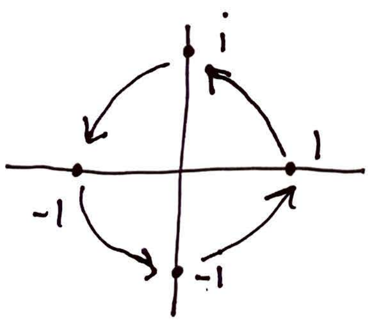
ⓑ $z \mapsto \underline{5}z$, $5 \in \mathbb{R}$, $0<5$ homothetic.

ⓒ $z \mapsto z+i-3 = z+(i-3)$ translation.

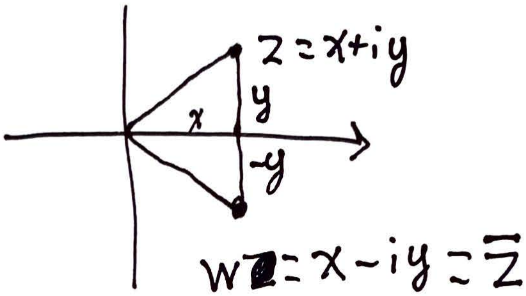
$= z+(-3+i)$

ⓓ $z \xrightarrow{f} iz = e^{i\pi/2} z$


$1e^{i\pi/2} = i$



# Reflections




Rem: <u>Reflection</u> about the $x$-axis is

$$z \mapsto \bar{z}.$$

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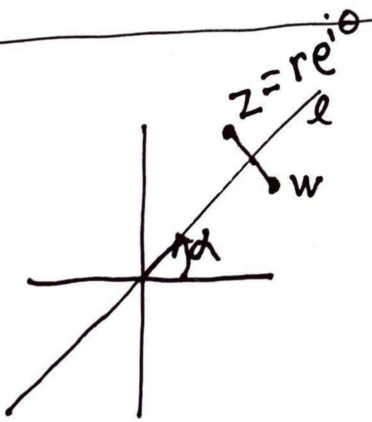
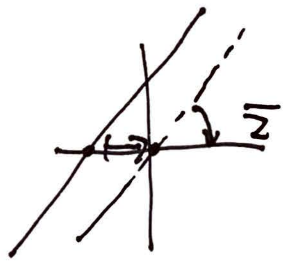



$$z \mapsto e^{-i\alpha} z \mapsto \overline{e^{-i\alpha} \cdot z} \mapsto e^{i\alpha} \cdot \overline{e^{-i\alpha} z}$$

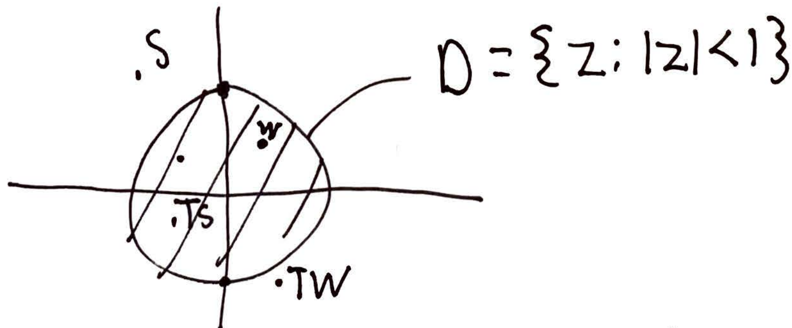
$$z \mapsto e^{i\alpha} \cdot \overline{e^{-i\alpha}} \cdot \bar{z} = e^{i\alpha} e^{i\alpha}\, \overline{re^{i\theta}} = e^{2i\alpha} re^{-i\theta}$$

$$z \mapsto re^{i(2\alpha - \theta)}$$

**Def:** A transformation is <u>Euclidean</u> if it is a translation, a rotation, ~~homothetic~~, or a reflection,

**Def:** A geometric transformation $z \mapsto \frac{1}{z}$ (for $z \neq 0$) is called an <u>inversion</u>.

<u>eg</u> $z = re^{i\theta} \xrightarrow{T} \frac{1}{z} = \frac{1}{re^{i\theta}} = \left(\frac{1}{r}\right)e^{-i\theta}$.




For $w \in D$, so $w = re^{i\theta}$ with $r < 1$

$Tw = T(re^{i\theta}) = \frac{1}{r}e^{-i\theta}$

For $s \notin D$ with $s = re^{i\theta}$ and $r > 1$

$Ts = T(re^{i\theta}) = \frac{1}{r}e^{-i\theta}$

Def: A transformation is <u>conformal</u> if it preserves angles.

Rem: all transformations we have listed are conformal.

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## <u>Mobius Transformations</u>

Let $a, b, c, d \in \mathbb{C}$ so that $ad - bc \neq 0$.

Def:

$$f_1(z) = z + \frac{d}{c} \qquad \text{translation}$$

$$f_2(z) = \frac{1}{z} \qquad \text{inversion}$$

$$f_3(z) = \frac{bc - ad}{c^2} z \qquad \text{homothetic + rotation}$$

$$f_4(z) = z + a/c \qquad \text{translation}$$

$$f(z) = f_4 \circ f_3 \circ f_2 \circ f_1(z)$$

$$f(z) = \frac{az + b}{cz + d}$$

Def: A <u>mobius transformation</u> is of the form

$$f(z) = \frac{az+b}{cz+d},$$

where $a,b,c,d \in \mathbb{C}$ with $ad-bc \neq 0$.