Elliptical Parallel axiom

For every line l and every point P not on l, there is no line m so that P is on m and $m \parallel l$.

Euclidean Parallel axiom

For every line l and every point P not on l, there is exactly one line m so that P is on m and $m \parallel l$.

Hyperbolic Parallel axiom

For every line l and every point P not on l, there are at least two lines m and n so that P is on both m and ~~m~~, n, and both m and n are parallel to l.

Def: Let S be a nonempty set. ~~and~~ A <u>transformation group</u> is a collection of transformations on S so that:

1. the identity transformation is in G;
2. all transformations in G are invertible, and their inverses are in G.
3. G is closed under composition.

Def: Let S be a non-empty set and G a transformation group on S, The pair (S, G) is called a <u>model of a geometry</u> or a <u>geometry</u>, the set S is the <u>underlying space</u> of the geometry, and the collection G is <u>transformation group</u> of the geometry.

Def: Let (S, G) be a geometry. If $A \subseteq S$, then we say that A is a figure of the geometry.

Def: Two figures A and B of a geometry (S, G) are congruent if there is a transformation $T \in G$ so that

$$T(A) = B.$$

$$T(A) = \{Tz : z \in A\}.$$

eg) Recall: Special Euclidean transformations are rotations and translations.

Euclidean transformations are rotations, translations, and reflections.

Let E_s be the collection of all Special Euclidean transformations and their compositions.

Let E be the collection of all Euclidean transformations and their compositions.

Eg| $(\mathbb{C}, E_S)$ is a geometry called the <u>Special Euclidean geometry</u>.

$(\mathbb{C}, E)$ is a geometry called <u>euclidean geometry</u>.

Def: Let (S, G) be a geometry, and D a set of figures in the geometry.

The set D is <u>invariant</u> if for every $A \in D$ and transformation $T \in G$,

$$T(A) \in D.$$

A function f defined on D is <u>invariant</u> if for every figure $B \in D$ and every transformation $T \in G$, $f(B) = f(T(B))$.

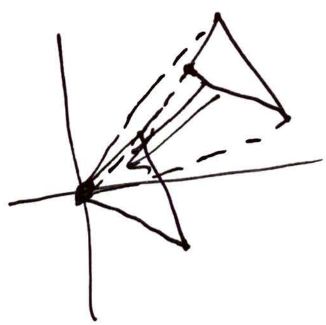
Def: Let ΔABC be a triangle. The <u>perimeter</u> of ΔABC is $AB + BC + AC$.

Eg Let D be the collection of all triangles in $(\mathbb{C}, E)$.

Define $p : D \to \mathbb{R}$ by $p(\Delta) =$ the perimeter of Δ.

p is an invariant function of $(\mathbb{C}, E)$.

Define $p' : D \to \mathbb{R}$ by $p'(\Delta)$ is the sum of the distances of each vertex of Δ to the origin.



p' is not invariant in $(\mathbb{C}, E)$.

Def: Two geometries (S_1, G_1) and (S_2, G_2) are <u>isomorphic</u> if there is a continuous invertible function $f: S_1 \to S_2$ so that

① for every $T_1 \in G_1$ there is a corresponding $T_2 \in G_2$ so that
$T_1 = f^{-1} \circ T_2 \circ f$

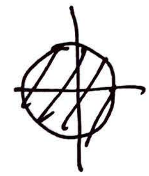
② for every $T_2' \in G_2$ there is a corresponding $T_1' \in G_1$ so that
$T_2' = f \circ T_1' \circ f^{-1}$.

The function f is called an <u>isomorphism</u>.

$$
\begin{tikzcd}
S_1 \arrow[r, "T_1"] \arrow[d, shift right, "f"'] & S_1 \arrow[d, shift left, "f"] \\
S_2 \arrow[r, "T_2"'] \arrow[u, shift right, "f^{-1}"'] & S_2 \arrow[u, shift left, "f^{-1}"]
\end{tikzcd}
$$

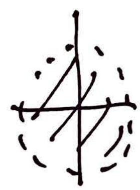
Def: The closed unit disk is

$$D = \{z : |z| \leq 1\}$$



The open unit disk is

$$D^{\circ} = \{z : |z| < 1\}$$

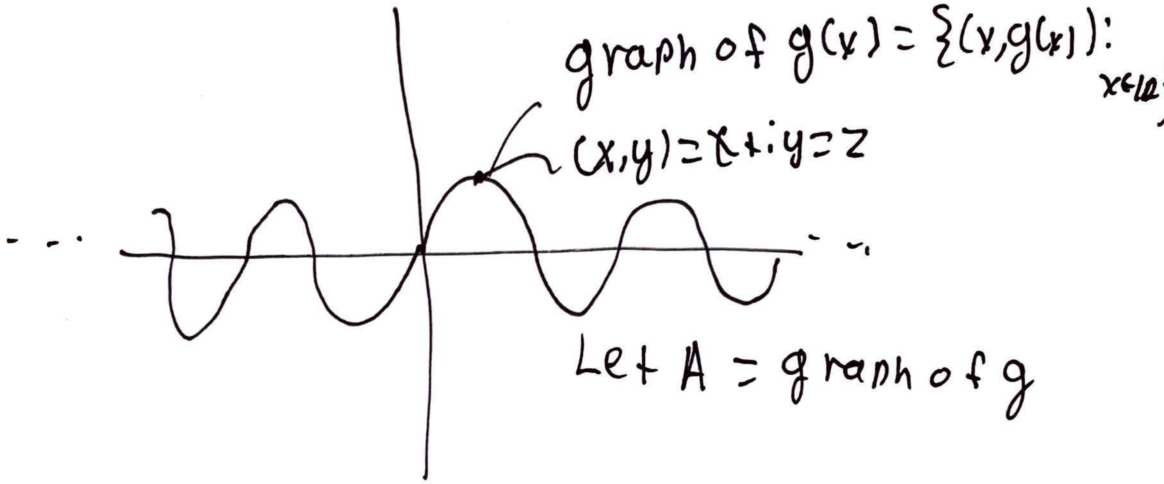


eg] Define $f : \mathbb{C} \to D^{\circ}$ by

$$f(z) = \frac{z}{\sqrt{1 + |z|^2}}.$$

f is continuous and invertible. So an isomorphism exists between $(\mathbb{C}, E)$ and (D°, E').

$g(x) = \sin(10x)$



Let A = graph of g

$f(A)$