

## Möbius Geometry

Def:  $\mathbb{C}^+ = \mathbb{C} \cup \{\infty\}$  equivalent to  $S^2$ .

Def: Let  $M$  be the set of all Möbius transformations, that is  $M$  is the set of transformations of the form

$$Tz = \frac{az + b}{cz + d},$$

for  $a, b, c, d \in \mathbb{C}$ , and  $ad - bc \neq 0$ .

The pair  $(\mathbb{C}^+, M)$  is a model of  
Möbius Geometry.

Thm: If  $T \in M$  is not the identity, Then  $T$  has either one or two fixed points

Pf: Let  $Tz = \frac{az + b}{cz + d}$ ,  $ad - bc \neq 0$ .

Case 1  $z$  is a fixed point of  $T$  if

$$z = Tz = \frac{az+b}{cz+d}$$

$$z(cz+d) = az+b$$

$$cz^2 + dz - az - b = 0$$

$$cz^2 + (d-a)z - b = 0$$

Case 1:  $c \neq 0$ . Then  $T$  has ~~at most 2 fixed~~ points.

Case 2:  $c=0$  Then  
 $a \neq d$

$$(d-a)z - b = 0$$

$$z = \frac{b}{d-a} \text{ is fixed}$$

$$z = \infty \text{ is fixed.}$$

Case 3:  $c=0$  Then  $-b=0$ , so

$$a=d$$

$$Tz = \frac{az}{d} = z, \text{ the id. } \square$$

Cor: A mobius transformation with three or more fixed points is the identity.

Thm: Fundamental Theorem of Mobius Geometry.

For any three distinct points

$z_1, z_2, z_3 \in \mathbb{C}^+$  and

three other distinct points

$w_1, w_2, w_3 \in \mathbb{C}^+$

there is a mobius transformation  $S_0$  that

$$z_1 \mapsto w_1$$

$$z_2 \mapsto w_2$$

$$z_3 \mapsto w_3.$$

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Pf: Let  $z_1, z_2, z_3 \in \mathbb{C}^+$  be distinct points  
and  $w_1, w_2, w_3 \in \mathbb{C}^+$  be distinct points.

$$\text{define } T_z = \frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)}$$

$$\text{Claim: } T_{z_1} = \cancel{0} \quad \checkmark$$

$$T_{z_2} = 0.$$

$$T_{z_3} = \cancel{\infty}$$

$$T_{z_1} = \frac{(z_1-z_2)(z_1-z_3)}{(z_1-z_3)(z_1-z_2)} = 1$$

$$T_{z_2} = \frac{(z_2-z_1)(z_1-z_3)}{(z_2-z_3)(z_1-z_2)} = 0$$

$$T_{z_3} = \frac{(z_3-z_1)(z_1-z_2)}{(z_3-z_2)(z_1-z_3)} = \cancel{\infty}$$

$$\text{Define } S_z = \frac{(z-w_2)(w_1-w_3)}{(z-w_3)(w_1-w_2)}$$

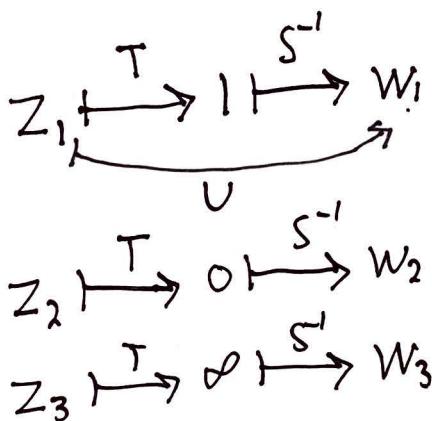
$$S_{w_1} = 1$$

$$S_{w_2} = 0$$

$$S_{w_3} = \cancel{\infty}$$

$$\begin{array}{ll} SW_1 = 1 & S_0 \quad W_1 = S^{-1}(1) \\ SW_2 = 0 & \quad \quad \quad W_2 = S^{-1}(0) \\ SW_3 = \infty & \quad \quad \quad W_3 = S^{-1}(\infty) \end{array}$$

Now define  $V = S^{-1} \circ T$ . Then



So there exists a  $U \in M$  with

$$Uz_1 = w_1$$

$$Uz_2 = w_2$$

$$Uz_3 = w_3$$

Uniqueness

Suppose  $V$  is a transformation with

$$Vz_1 = w_1$$

$$Vz_2 = w_2$$

$$Vz_3 = w_3$$

Then  $(V^{-1} \circ V)z_1 = V^{-1}(Vz_1) = V^{-1}(w_1) = z_1$

$$V^{-1}Vz_2 = z_2$$

$$V^{-1}Vz_3 = z_3$$

So  $V^{-1}V = \text{Id}$ , since it has 3 fixed points.

Hence  $V = V$ .

Def: The cross ratio of 4 complex numbers  $z_0, z_1, z_2$  and  $z_3$  is

$$(z_0, z_1, z_2, z_3) = \frac{(z_0 - z_1)(z_1 - z_3)}{(z_0 - z_3)(z_1 - z_2)}$$

Def: A cline is a circle or a line.

Rem: On  $S^2$  a line is a circle thru the North pole.

Thm: A mobius transformation takes  
clines to clines.

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Next time: Work out examples  
Post HW.