Mobius Geometry

Def: $\mathbb{C}^+ = \mathbb{C} \cup \{\infty\}$ equivalent to S^2.

Def: Let M be the set of all Mobius transformations, that is M is the set of transformations of the form

$$Tz = \frac{az+b}{cz+d},$$

for $a, b, c, d \in \mathbb{C}$, and $ad-bc \neq 0$.

The pair $(\mathbb{C}^+, M)$ is a model of <u>Mobius Geometry</u>.

Thm: If $T \in M$ is not the identity, Then T has either one or two fixed points

Pf: Let $Tz = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$.

~~Case 1~~ z is a fixed point of T if

$$z = Tz = \frac{az+b}{cz+d}$$

$$z(cz+d) = az+b$$

$$cz^2 + dz - az - b = 0$$

$$cz^2 + (d-a)z - b = 0$$

Case 1: $c \neq 0$. Then T has ~~at most~~ 2 fixed points.

Case 2: $c = 0$ Then
$a \neq d$

$$(d-a)z - b = 0$$

$z = \frac{b}{d-a}$ is fixed

$z = \infty$ is fixed.

Case 3: $c = 0$ Then $-b = 0$, so
$a = d$

$Tz = \frac{az}{d} = z$, the id. $\square$

Cor: A mobius Transformation with three or more fixed points is the identity.

Thm: Fundamental Theorem of Mobius Geometry.

For any three distinct points

$z_1, z_2, z_3 \in \mathbb{C}^+$ and

three other distinct points

$w_1, w_2, w_3 \in \mathbb{C}^+$

there is a mobius transformation so that

$$z_1 \mapsto w_1$$
$$z_2 \mapsto w_2$$
$$z_3 \mapsto w_3.$$

Pf: Let $z_1, z_2, z_3 \in \mathbb{C}^+$ be distinct points and $w_1, w_2, w_3 \in \mathbb{C}^+$ be distinct points.

define $Tz = \dfrac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}$.

claim: $Tz_1 = 1$ ✓

$Tz_2 = 0.$

$Tz_3 = \infty$

$$Tz_1 = \frac{\cancel{(z_1 - z_2)}\cancel{(z_1 - z_3)}}{\cancel{(z_1 - z_3)}\cancel{(z_1 - z_2)}} = 1$$

$$Tz_2 = \frac{(z_2 - z_2)(z_1 - z_3)}{(z_2 - z_3)(z_1 - z_2)} = 0$$

$$Tz_3 = \frac{(z_3 - z_2)(z_1 - z_3)}{(z_3 - z_3)(z_1 - z_2)} = \infty$$

Define $Sz = \dfrac{(z - w_2)(w_1 - w_3)}{(z - w_3)(w_1 - w_2)}$

$Sw_1 = 1$

$Sw_2 = 0$

$Sw_3 = \infty$

$$Sw_1 = 1 \qquad \text{So} \qquad w_1 = S^{-1}(1)$$
$$Sw_2 = 0 \qquad\qquad\qquad w_2 = S^{-1}(0)$$
$$Sw_3 = \infty \qquad\qquad\qquad w_3 = S^{-1}(\infty)$$

Now define $U = S^{-1} \circ T$. Then

$$\begin{tikzcd} z_1 \arrow[r, mapsto, "T"] \arrow[rr, mapsto, bend right, "U"'] & 1 \arrow[r, mapsto, "S^{-1}"] & w_1 \end{tikzcd}$$

$$z_2 \xmapsto{T} 0 \xmapsto{S^{-1}} w_2$$

$$z_3 \xmapsto{T} \infty \xmapsto{S^{-1}} w_3$$

So there exists a $U \in M$ with

$$Uz_1 = w_1$$
$$Uz_2 = w_2$$
$$Uz_3 = w_3$$

Uniqueness

Suppose V is a transformation with

$$Vz_1 = w_1$$
$$Vz_2 = w_2$$
$$Vz_3 = w_3$$

Then $(V^{-1} \circ U) z_1 = V^{-1}(U z_1) = V^{-1}(w_1) = z_1$

$$V^{-1} U z_2 = z_2$$

$$V^{-1} U z_3 = z_3$$

So $V^{-1} U = Id$, since it has 3 fixed points.

Hence $U = V$.

Def: The <u>cross ratio</u> of 4 complex numbers z_0, z_1, z_2 and z_3 is

$$(z_0, z_1, z_2, z_3) = \frac{(z_0 - z_1)(z_2 - z_3)}{(z_0 - z_3)(z_2 - z_1)}$$

Def: A <u>cline</u> is a circle or a line.

Rem: On S^2 a line is a circle thru the north pole.

Thm: A mobius transformation takes
clines to clines.

Next time: Work out examples
Post HW.