Hyperbolic Geometry

Thm:

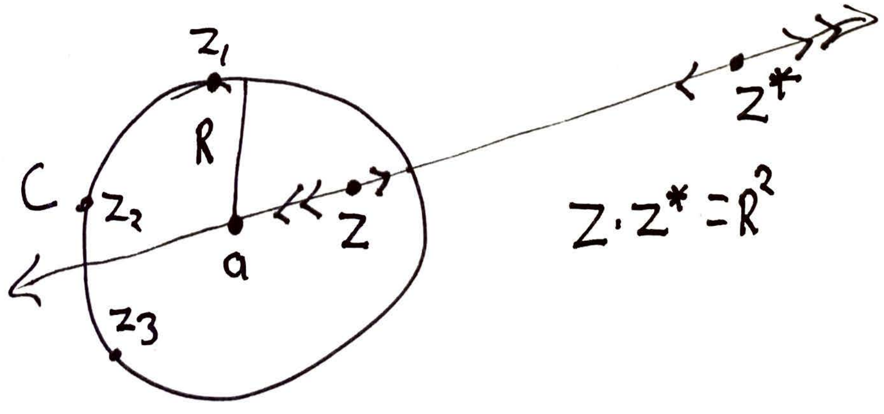
① Let z_0, z_1, z_2, z_3 be four distinct points in $\mathbb{C}$, and T a Mobius transformation. Then

$$(z_0, z_1, z_2, z_3) = (Tz_0, Tz_1, Tz_2, Tz_3).$$

② ~~The~~ Theorem: The cross ratio (z_0, z_1, z_2, z_3) is real if and only if z_0, z_1, z_2, z_3 all lie on a cline.

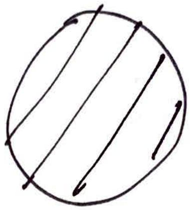
Def: Let C be cline containing 3 distinct points z_1, z_2 and z_3. Two points z and z^* are symmetric with respect to C if

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}.$$



$$z \cdot z^* = R^2$$

Recall that the unit disk is $D = \{z \in \mathbb{C} : |z| < 1\}$.



Prop: If T is Mobius transformation and $T(D) = D$, then T is of the form

$$Tz = e^{i\theta}\left(\frac{z - z_0}{1 - \bar{z}_0 z}\right)$$

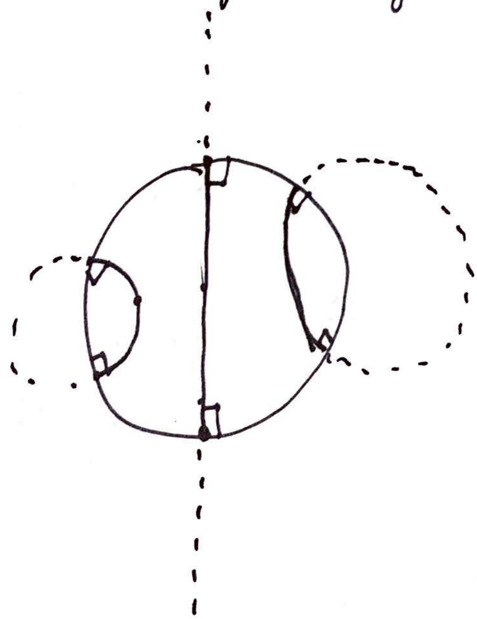
for some $\theta \in \mathbb{R}$ and some $z_0 \in \mathbb{C}$ with $|z_0| < 1$.

Def: Let $\mathcal{H}$ be the set of mobius transformations mapping D to D, that is

$$\mathcal{H} = \left\{ z \mapsto e^{i\theta}\left(\frac{z - z_0}{1 - \bar{z}_0 z}\right) : \theta \in \mathbb{R}, z_0 \in D \right\}.$$

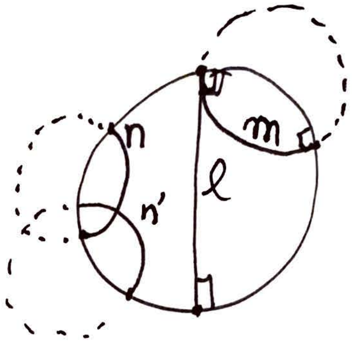
The pair $(D, \mathcal{H})$ is a model for hyperbolic geometry. D is called the hyperbolic plane, and $\mathcal{H}$ is the hyperbolic group.

Def: A <u>hyperbolic strait line</u> is a Euclidean circle or line in $\mathbb{C}$ that intersects the unit circle at right angles.



Thm: In hyperbolic geometry, all hyperbolic strait lines are congruent. Two points in the hyperbolic plane determine a unique hyperbolic strait line.

Idea: Parallel lines don't intersect.



Def: Points on the unit circle are called ideal points.

Def: Two hyperbolic lines are parallel if they don't intersect in D, but share an ideal point.

Two hyperbolic lines are hyperparallel if they don't intersect in D, and don't have an ideal point in common.